Chapter 19

PROPERTIES OF THE FREQUENCY-MODULATED SIGNAL

Phase and frequency modulation are defined, and the similarities and differences between these two forms of angle modulation are discussed. The expression for the FM or PM signal is analyzed to determine the spectrum when the modulating signal consists of one or more sinusoids. The problem of extending this analysis to cover more complex modulating signals is then considered. Other topics include the bandwidth required for transmission, the effect of nonlinear input-output characteristics, and limiters.

The material presented in this chapter is essentially a review of certain aspects of modulation theory which is necessary as background for those chapters on FM systems analysis which follow. For a more detailed discussion of particular points, the reader is referred to standard texts on modulation theory, such as those listed under References at the end of this chapter.

INTRODUCTION TO PHASE AND FREQUENCY MODULATION

Comparison of Amplitude Modulation and Angle Modulation

Any sinusoidal carrier may be subjected to two distinctly different types of modulation. These are amplitude modulation and angle modulation, both of which may be defined with reference to Eq. (19-1).

$$M(t) = A \cos (\omega_c t + \phi) \qquad (19\text{-}1)$$

Here A is the amplitude of the sinusoidal carrier and $\omega_c t + \phi$ is the angle. The carrier frequency is ω_c radians/second. If the coefficient A is by some means varied with time, amplitude modulation is obtained.

If, instead, ϕ is varied with time, the result is angle modulation. The general angle-modulated wave might then be expressed as

$$M(t) = A_c \cos [\omega_c t + \phi(t)] \qquad (19\text{-}2)$$

where

$M(t)$ = angle-modulated carrier

A_c = peak carrier amplitude in volts

ω_c = carrier frequency in radians/second

$\phi(t)$ = angle modulation in radians

If angle modulation is used to transmit information, it is necessary that $\phi(t)$ be a prescribed function of the modulating signal to be transmitted. For example, if $V(t)$ is the modulating signal, the angle modulation $\phi(t)$ can be expressed mathematically as

$$\phi(t) = f[V(t)] \qquad (19\text{-}3)$$

Many varieties of angle modulation are possible, depending on the selection of the functional relationship. Two of these are important enough to have the individual names of *phase modulation* and *frequency modulation.*

Phase Modulation and Frequency Modulation

The difference between phase and frequency modulation can be understood by first defining four terms, as follows.

The instantaneous phase and instantaneous phase deviation are, with reference to Eq. (19-2),

$$\textit{Instantaneous phase} = \omega_c t + \phi(t) \text{ radians} \qquad (19\text{-}4)$$

and

$$\textit{Instantaneous phase deviation} = \phi(t) \text{ radians} \qquad (19\text{-}5)$$

The instantaneous frequency of an angle-modulated carrier is defined as the first time derivative of the instantaneous phase. In terms of Eq. (19-2) the instantaneous frequency and the instantaneous frequency deviation are

$$\begin{aligned}\textit{Instantaneous frequency} &= \frac{d}{dt}[\omega_c t + \phi(t)] \\ &= \omega_c + \phi'(t) \text{ radians/second} \qquad (19\text{-}6)\end{aligned}$$

and

$$\textit{Instantaneous frequency deviation} = \phi'(t) \text{ radians/second} \tag{19-7}$$

Using these definitions, phase modulation (PM) can be defined as angle modulation in which the instantaneous phase deviation, $\phi(t)$, is proportional to the modulating signal, $V(t)$. Similarly, frequency modulation (FM) is angle modulation in which the instantaneous frequency deviation, $\phi'(t)$, is proportional to the modulating signal, $V(t)$. Mathematically, these statements become, for phase modulation,

$$\phi(t) = kV(t) \tag{19-8}$$

and for frequency modulation,

$$\phi'(t) = k_1 V(t) \tag{19-9}$$

from which

$$\phi(t) = k_1 \int V(t)\,dt, \tag{19-10}$$

where k and k_1 are constants.

These results are summarized in Table 19-1. This table also illustrates phase-modulated and frequency-modulated waves which occur when the modulating wave is a single sinusoid.

TABLE 19-1. EQUATIONS FOR PHASE- AND FREQUENCY-MODULATED CARRIERS

Type of modulation	Modulating signal	Angle-modulated carrier
(a) Phase	$V(t)$	$M(t) = A_c \cos [\omega_c t + kV(t)]$
(b) Frequency	$V(t)$	$M(t) = A_c \cos [\omega_c t + k_1 \int V(t)dt]$
(c) Phase	$A_v \cos \omega_v t$	$M(t) = A_c \cos (\omega_c t + kA_v \cos \omega_v t)$
(d) Frequency	$-A_v \sin \omega_v t$	$M(t) = A_c \cos \left(\omega_c t + \frac{k_1 A_v}{\omega_v} \cos \omega_v t \right)$
(e) Frequency	$A_v \cos \omega_v t$	$M(t) = A_c \cos \left(\omega_c t + \frac{k_1 A_v}{\omega_v} \sin \omega_v t \right)$

Figure 19-1 illustrates amplitude, phase, and frequency modulation of a carrier by a signal which consists of a single sinusoid. The similarity of waveforms of the PM and FM waves shows that for angle-modulated waves it is necessary to know the modulation function; that is, the waveform alone cannot be used to distinguish be-

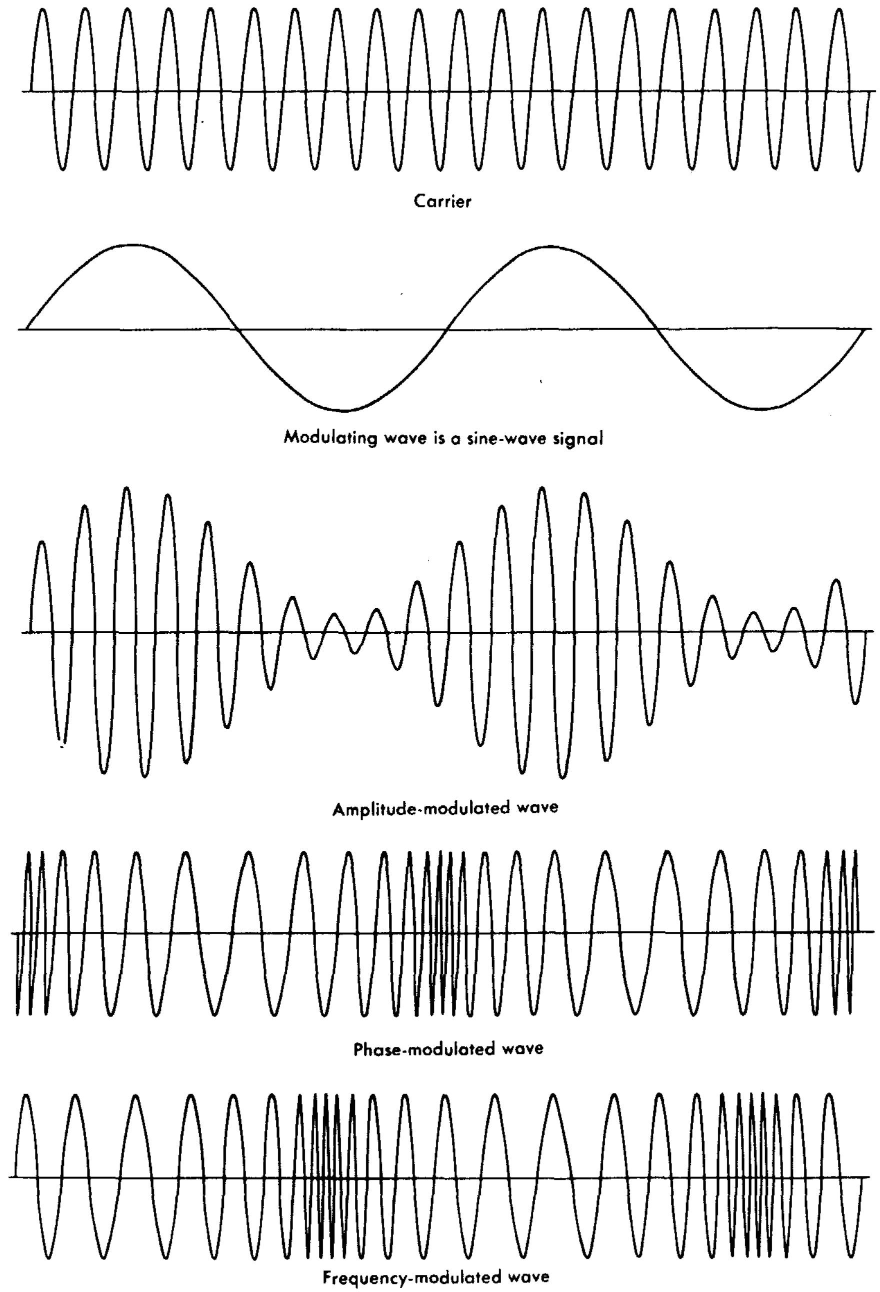


FIG. 19-1. Amplitude, phase, and frequency modulation of a sine-wave carrier by a sine-wave signal.

tween PM and FM. Similarly, one cannot look at Eq. (19-2) and tell whether it represents an FM or a PM wave. It could be either. A knowledge of the modulation function, however, will permit the correct identification. If $\phi(t) = kV(t)$, it is phase modulation and, if $\phi'(t) = k_1 V(t)$, it is frequency modulation.

Comparison of (c), (d), and (e) in Table 19-1 shows that the expression for a carrier which is phase or frequency modulated by a sinusoidal type signal can be written in the general form of

$$M(t) = A_c \cos(\omega_c t + X \cos \omega_v t) \tag{19-11}$$

where

$$X = kA_v \qquad \text{for PM,} \tag{19-12}$$

and

$$X = \frac{k_1 A_v}{\omega_v} \qquad \text{for FM} \tag{19-13}$$

Here X is the peak phase deviation in radians and is called the *index of modulation.* For PM the index of modulation is a constant independent of the frequency of the modulating wave, and for FM it is inversely proportional to the frequency of the modulating wave. Note that in the FM case, the modulation index can also be expressed as the peak frequency deviation, k_1A_v, divided by the modulating signal frequency, ω_v. The terms *high-index* and *low-index* of modulation are often used. It is difficult to define a sharp division. In general, however, the term *low-index* would normally be used when the peak phase deviation is less than one radian. In a later section, the effect of the index of modulation on the frequency spectrum of the modulated wave will be considered.

When the modulation function consists of a single sinusoid, it is evident from Eq. (19-11) that the phase angle of the carrier varies from its unmodulated value in a simple sinusoidal fashion, with the peak phase deviation being equal to X. The phase deviation can also be expressed in terms of the rms phase deviation, which for this case is $X/\sqrt{2}$. Similarly, the frequency deviation of a sinusoidally modulated carrier can be expressed either in terms of the peak frequency deviation, k_1A_v, or the rms frequency deviation, which is $k_1A_v/\sqrt{2}$. In the more general case where the modulation function is a complex signal, such as broadband telephone multiplex or noise, the peak value of the modulation function, and hence the peak phase or frequency deviation, is not a precisely defined magnitude. For these cases, the phase or frequency deviation of the carrier is normally expressed in terms of the rms value. The peak phase or frequency deviation is then equal to the rms value times an appropriate peak factor for the modulating signal, with the peak factor chosen so that the actual deviation exceeds the computed peak deviation only a specified percentage of the time. The determination of the peak factor and the general problem of treating these complex signals are discussed in Chap. 10.

Phase and Frequency Modulators and Demodulators

A phase modulator, designated here as a PM modulator, is a device which varies the phase of a carrier so that the instantaneous phase deviation is proportional to the modulating wave. On the other hand an FM modulator, often referred to as an *FM deviator,* produces an instantaneous phase deviation proportional to the integral of the modulating wave. This suggests the following possibility. If a modulating wave $V(t)$ is differentiated before being applied to an FM modulator, the instantaneous phase deviation will be proportional to the integral of $V'(t)$, or, in other words, proportional to $V(t)$. Thus, an FM modulator that is preceded by a differentiator actually produces an instantaneous phase deviation proportional to the modulating wave and is therefore equivalent to a PM modulator.

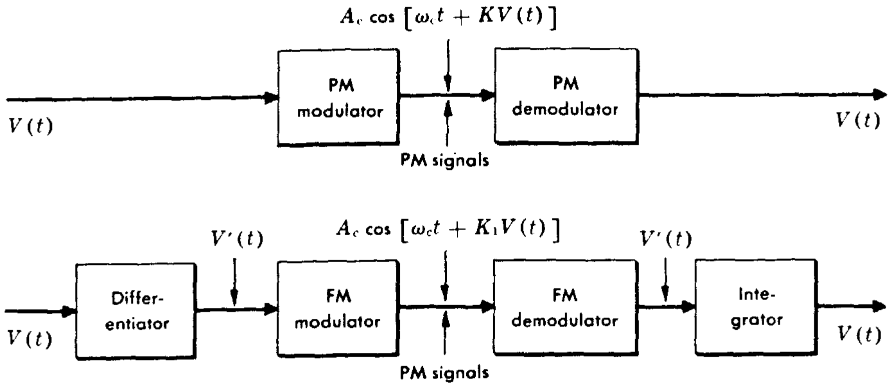
Other equivalences are also possible. For example, a PM demodulator is equivalent to an FM demodulator, commonly called an *FM discriminator,* followed by an integrator. Several equivalences are listed below and illustrated in Fig. 19-2.

1. PM modulator = differentiator + FM modulator
2. PM demodulator = FM demodulator + integrator
3. FM modulator = integrator + PM modulator
4. FM demodulator = PM demodulator + differentiator

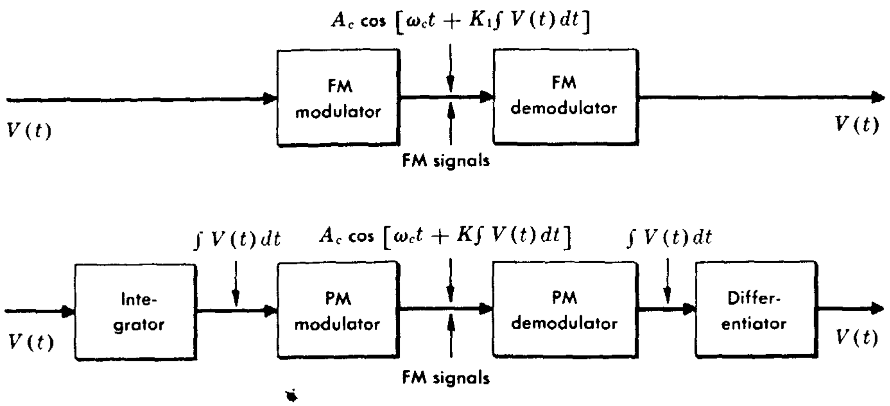
SPECTRA OF FM AND PM WAVES

Introduction to Frequency Analysis of FM and PM Waves

In the case of amplitude modulation, it is easy to demonstrate that the frequency components of the modulated wave consist of a carrier, an upper sideband, and a lower sideband. The frequency components of the upper sideband have the same form as the components of the modulating wave except that they have been translated upward in frequency by an amount equal to the carrier frequency. The lower sideband is a mirror image of the upper sideband about the carrier frequency. This is illustrated in Fig. 19-3. For every component at



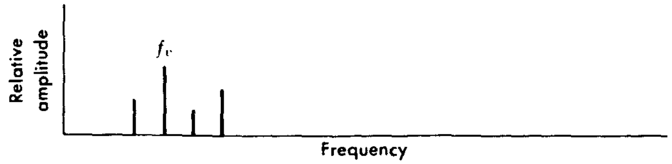
(a) Two methods of phase modulation and demodulation



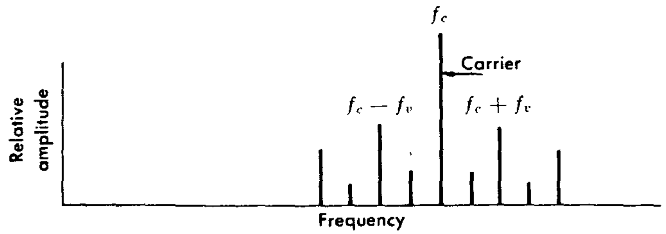
(b) Two methods of frequency modulation and demodulation

FIG. 19-2. Phase and frequency modulation and demodulation.

a frequency f_v in the modulating wave there are two components in the modulated wave; one is at a frequency $f_c + f_v$ and one at a frequency $f_c - f_v$, where f_c is the carrier frequency. In a sense then, superposition holds since the effect produced by any particular modulating component does not depend on the other modulating components which are present. This makes amplitude modulation easy to deal with. For example, the bandwidth required to transmit a double-sideband AM wave is easily determined. If the highest frequency component in the modulating wave is f_b, the modulated wave is re-



(a) Frequency components of modulating wave



(b) Frequency components of modulated wave

FIG. 19-3. Frequency spectrum of an amplitude-modulated wave.

stricted to the frequency range which extends from $f_c - f_b$ to $f_c + f_b$, and the required bandwidth is $2f_b$ centered at a frequency f_c.

In the case of frequency modulation, the frequency components of the modulated wave are much more complexly related to the components in the modulating wave. In a strict mathematical sense, a single modulating tone produces an infinity of sideband tones, although most are negligibly small. This in itself complicates the frequency spectrum of an FM wave. In addition, the sideband components produced by any single-frequency component in the modulating wave depend on all the frequency components in the modulating wave. Hence, superposition does not apply.

Is it really advantageous to deal with the frequency components of an FM wave in view of this difficulty? At the present time, the answer seems to be that this is the best way known. The transmission characteristics of networks, interstages, and other transmission paths are specified as a function of frequency. Imperfect transmission at any particular frequency will affect only those frequency components of the signal which are at that frequency, but this in turn may cause serious impairment to the signal being transmitted if the imperfection is not properly equalized before the FM discriminator. Furthermore, consider the problem of determining the required bandwidth; this clearly depends on the location of all of the important frequency components in the wave. So in spite of the difficulty, some knowledge of the frequency components of an FM signal is essential.

Phase and Frequency Modulation by a One-Tone Signal

The frequency analysis of the FM or PM wave will now be considered for the case where the modulating signal is a single sinusoid. Let the frequency of the modulating signal be ω_1 radians per second and the peak phase deviation be X_1 radians. Thus, for this case, $M(t)$ can be written as

$$M(t) = A_c \cos(\omega_c t + X_1 \cos \omega_1 t) \tag{19-14}$$

As this equation now stands, the separate frequency components are not obvious. However, Bessel function identities are available which may be applied directly to the problem at hand; several are given below.

$$\sin(\alpha + X \sin \beta) = \sum_{n=-\infty}^{\infty} J_n(X) \sin(\alpha + n\beta) \tag{19-15}$$

$$\cos(\alpha + X \sin \beta) = \sum_{n=-\infty}^{\infty} J_n(X) \cos(\alpha + n\beta) \tag{19-16}$$

$$\sin(\alpha + X \cos \beta) = \sum_{n=-\infty}^{\infty} J_n(X) \sin\left(\alpha + n\beta + \frac{n\pi}{2}\right) \tag{19-17}$$

$$\cos(\alpha + X \cos \beta) = \sum_{n=-\infty}^{\infty} J_n(X) \cos\left(\alpha + n\beta + \frac{n\pi}{2}\right) \tag{19-18}$$

Here $J_n(X)$ is the Bessel function of the first kind of nth order and of argument X. Values of $J_n(X)$ for several values of X are shown in Table 19-2. A more complete tabulation of values may be obtained in References 2 and 3. Note that the argument X is the index of modulation.

The identity given by Eq. (19-18) can be applied to the signal of Eq. (19-14) to give

$$M(t) = A_c \sum_{n=-\infty}^{\infty} J_n(X_1) \cos\left(\omega_c t + n\omega_1 t + \frac{n\pi}{2}\right) \tag{19-19}$$

The first few terms may be written as

$$\begin{aligned} M(t) = A_c \Big\{ & J_0(X_1) \cos \omega_c t + J_1(X_1) \cos\left[(\omega_c + \omega_1)t + \frac{\pi}{2}\right] \\ & + J_{-1}(X_1) \cos\left[(\omega_c - \omega_1)t - \frac{\pi}{2}\right] \\ & + J_2(X_1) \cos\left[(\omega_c + 2\omega_1)t + \frac{2\pi}{2}\right] \\ & + J_{-2}(X_1) \cos\left[(\omega_c - 2\omega_1)t - \frac{2\pi}{2}\right] + \cdots \Big\} \end{aligned} \tag{19-20}$$

Because of the identity

$$J_{-n}(X) = (-1)^n J_n(X) \tag{19-21}$$

it follows that $M(t)$ can be written as

$$\begin{aligned} M(t) = A_c \Big\{ & J_0(X_1) \cos \omega_c t + J_1(X_1) \cos\left[(\omega_c + \omega_1)t + \frac{\pi}{2}\right] \\ & + J_1(X_1) \cos\left[(\omega_c - \omega_1)t + \frac{\pi}{2}\right] - J_2(X_1) \cos[(\omega_c + 2\omega_1)t] \\ & - J_2(X_1) \cos[(\omega_c - 2\omega_1)t] + \cdots \Big\} \end{aligned} \tag{19-22}$$

Equation (19-22) shows that the single sinusoidal modulating wave has produced sets of sidebands displaced from the carrier by multiples of the modulating frequency. These successive sets of sidebands are often referred to as first order sidebands, second order sidebands, etc., the magnitudes of which, relative to the carrier, are determined by the coefficients $J_1(X_1)$, $J_2(X_1)$, etc., respectively. As Table 19-2 and Fig. 19-4 show, the higher order sidebands rapidly become unimportant as the index of modulation, X, becomes less than unity. For larger values of X, the value of $J_n(X)$ starts to decrease rapidly as soon as $n = X$.

TABLE 19-2. VALUES OF $J_n(X)$ FOR SEVERAL VALUES OF X

	$X = 1/2$	$X = 1$	$X = 2$	$X = 3$	$X = 10$
$J_0(X)$	0.938	0.765	0.224	−0.260	−0.246
$J_1(X)$	0.242	0.440	0.577	0.339	0.043
$J_2(X)$	0.031	0.115	0.353	0.486	0.255
$J_3(X)$	0.003	0.020	0.129	0.309	0.058
$J_4(X)$	0.000	0.002	0.034	0.132	−0.220

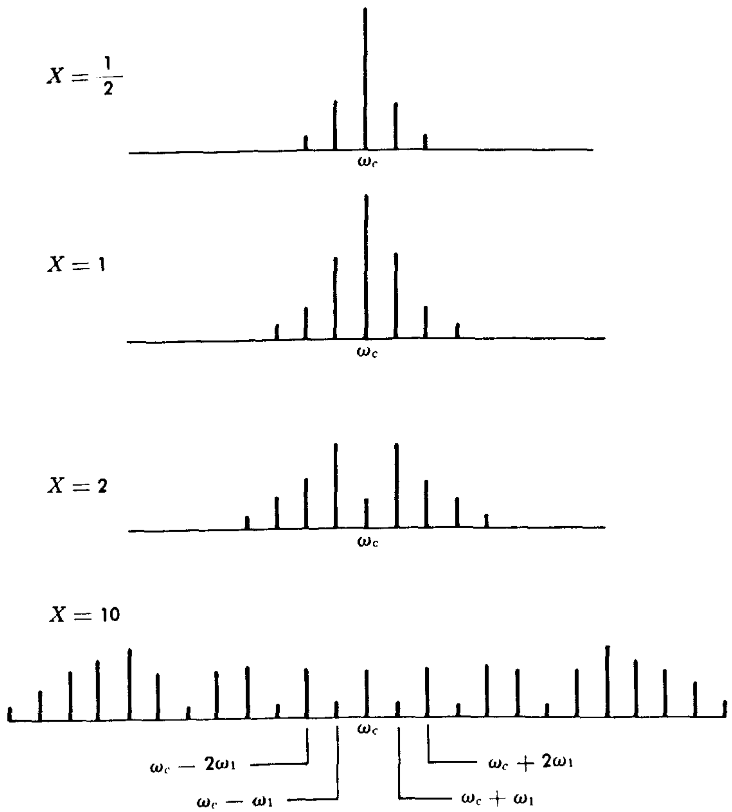


FIG. 19-4. Amplitude spectrum of $A_c \cos(\omega_c t + X \cos \omega_1 t)$ for various values of X.

As the index, X, is increased from small values, the magnitude of the carrier, given by $A_c J_0(X)$, decreases. For $X = 2.405$, $J_0(X) = 0$, and the carrier vanishes. This property is frequently used to determine, or to adjust, the deviation sensitivity [the coefficient k_1 of Eq. (19-13)] of an FM deviator.

In many situations, useful engineering results can be obtained by replacing the Bessel functions by their power series expansions. As may be seen from Table 19-3, the power series expansions converge relatively swiftly, provided that the index of modulation is less than unity.

TABLE 19-3. BESSEL FUNCTION SERIES EXPANSIONS

Function	Expansion
$J_0(X)$	$1 - \frac{X^2}{4} + \frac{X^4}{64} - \cdots$
$J_1(X)$	$\frac{X}{2} - \frac{X^3}{16} + \frac{X^5}{384} - \cdots$
$J_2(X)$	$\frac{X^2}{8} - \frac{X^4}{96} + \cdots$
$J_3(X)$	$\frac{X^3}{48} - \frac{X^5}{768} + \cdots$

For single-tone modulation, substitution of the series expansions of Table 19-3 into Eq. (19-22) yields the expressions for magnitudes of the spectral components, relative to the carrier, given in Table 19-4. This table also gives numerical values for $X = 1/2$ and $X = 1$, which may be compared to the exact values of Table 19-2.

TABLE 19-4. APPROXIMATE EXPRESSIONS FOR AMPLITUDES OF SPECTRAL COMPONENTS (RELATIVE TO CARRIER) FOR SINGLE-TONE MODULATION

Component	Approximate expression	Numerical value	
		$X_1 = 1/2$	$X_1 = 1$
$J_0(X_1)$ (carrier)	$1 - \frac{X_1^2}{4}$	0.938	0.750
$J_1(X_1)$ (first order sideband)	$\frac{X_1}{2}\left(1 - \frac{X_1^2}{8}\right)$	0.242	0.438
$J_2(X_1)$ (second order sideband)	$\frac{X_1^2}{8}$	0.031	0.125
$J_3(X_1)$ (third order sideband)	$\frac{X_1^3}{48}$	0.003	0.021

Phase and Frequency Modulation by a Two-Tone Signal

The determination of the spectral components of an FM or PM wave becomes increasingly difficult as the number of modulating tones is increased. To develop the concepts and analysis involved, it is useful to consider next the case where a second sinusoidal signal is added to the modulating signal. For this case the modulated carrier can be written as

$$M(t) = A_c \cos\,(\omega_c t + X_1 \cos \omega_1 t + X_2 \cos \omega_2 t) \tag{19-23}$$

It is convenient to start the analysis by first writing Eq. (19-23) in the form

$$M(t) = A_c \cos\left(\frac{\omega_c t}{2} + X_1 \cos \omega_1 t + \frac{\omega_c t}{2} + X_2 \cos \omega_2 t\right) \tag{19-24}$$

Using the trigonometric identity

$$\cos\,(A + B) = \cos A \cos B - \sin A \sin B, \tag{19-25}$$

Eq. (19-24) can be written as

$$\begin{aligned} M(t) = A_c \cos\left(\frac{\omega_c t}{2} + X_1 \cos \omega_1 t\right) &\cos\left(\frac{\omega_c t}{2} + X_2 \cos \omega_2 t\right) \\ - \sin\left(\frac{\omega_c t}{2} + X_1 \cos \omega_1 t\right) &\sin\left(\frac{\omega_c t}{2} + X_2 \cos \omega_2 t\right) \end{aligned} \tag{19-26}$$

The identities of Eqs. (19-15) through (19-18) can now be applied; thus,

$$\begin{aligned} M(t) = A_c \Bigg\{ &\Bigg[\sum_{n=-\infty}^{\infty} J_n(X_1) \cos\left(\frac{\omega_c t}{2} + n\omega_1 t + \frac{n\pi}{2}\right) \\ &\cdot \sum_{m=-\infty}^{\infty} J_m(X_2) \cos\left(\frac{\omega_c t}{2} + m\omega_2 t + \frac{m\pi}{2}\right)\Bigg] \\ &- \Bigg[\sum_{n=-\infty}^{\infty} J_n(X_1) \sin\left(\frac{\omega_c t}{2} + n\omega_1 t + \frac{n\pi}{2}\right) \\ &\cdot \sum_{m=-\infty}^{\infty} J_m(X_2) \sin\left(\frac{\omega_c t}{2} + m\omega_2 t + \frac{m\pi}{2}\right)\Bigg]\Bigg\} \end{aligned} \tag{19-27}$$

This equation may be written as

$$\begin{aligned} M(t) = A_c \Bigg\{ &\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(X_1)\, J_m(X_2) \\ &\cdot \Bigg[\cos\left(\frac{\omega_c t}{2} + n\omega_1 t + \frac{n\pi}{2}\right) \cos\left(\frac{\omega_c t}{2} + m\omega_2 t + \frac{m\pi}{2}\right) \\ &- \sin\left(\frac{\omega_c t}{2} + n\omega_1 t + \frac{n\pi}{2}\right) \sin\left(\frac{\omega_c t}{2} + m\omega_2 t + \frac{m\pi}{2}\right)\Bigg]\Bigg\} \end{aligned} \tag{19-28}$$

By means of Eq. (19-25) further reduction can be obtained to give

$$\begin{aligned} M(t) = A_c &\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(X_1)\, J_m(X_2) \\ &\cdot \cos\left[(\omega_c + n\omega_1 + m\omega_2)\, t + \frac{(n+m)\pi}{2}\right] \end{aligned} \tag{19-29}$$

This equation is the desired result. It indicates that not only will there be sideband components displaced from the carrier by all possible multiples of the individual modulating frequencies, but also there will be components displaced by all possible sums and differences of multiples of the modulating frequencies. The sideband components in Eq. (19-29) can be split into three types: (1) the frequencies which would have been present (as in Fig. 19-4) if only $X_1 \cos \omega_1 t$ had been applied as a modulating signal; (2) the frequencies which would have been present if only $X_2 \cos \omega_2 t$ had been applied; and (3) all the possible sum and difference components of the form $(\omega_c \pm n\omega_1 \pm m\omega_2)$.

Figure 19-5 shows the amplitude and relative phase spectra of the zero, first, and second order components obtained from Eq. (19-29) for the condition $X_1 = 1/2$ and $X_2 = 1$. In the general case, the order of the component is equal to the sum of the magnitudes of the orders of the Bessel functions used to compute the amplitude of that component. For example, a second order component in Eq. (19-29) is any component for which $|m| + |n| = 2$.

An important difference between frequency modulation and amplitude modulation can be illustrated if in Eq. (19-23), for the two-tone FM wave, the modulation function for the second tone is written

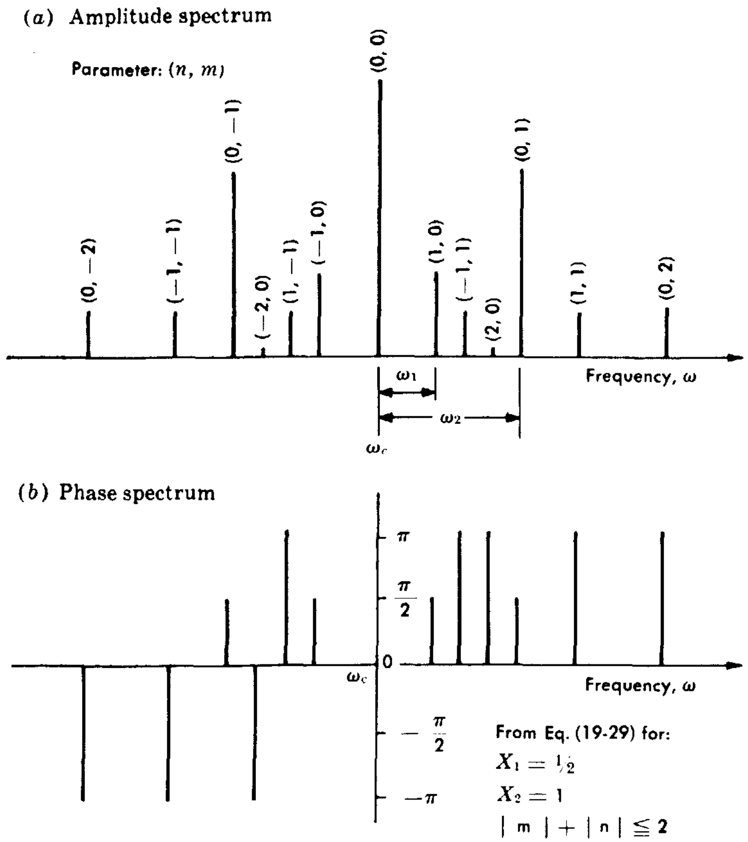


FIG. 19-5. Amplitude and relative phase spectra.

as $X_2 \sin \omega_2 t$ instead of $X_2 \cos \omega_2 t$. Equation (19-23) then becomes

$$M(t) = A_c \cos(\omega_c t + X_1 \cos \omega_1 t + X_2 \sin \omega_2 t) \qquad (19\text{-}30)$$

For this case, the reduced equation corresponding to Eq. (19-29) may be obtained by substituting $\omega_2 t - \pi/2$ for $\omega_2 t$ in Eq. (19-29), which yields

$$M(t) = A_c \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(X_1)\, J_m(X_2) \cdot \cos\left[(\omega_c + n\omega_1 + m\omega_2)t + \frac{n\pi}{2}\right] \qquad (19\text{-}31)$$

From this double summation, certain terms are extracted and listed in Table 19-5.

TABLE 19-5. SELECTED TERMS FROM EQ. (19-31)

Sideband	n	m	Term
(a) Upper second order sideband of ω_1	2	0	$A_c J_2(X_1) J_0(X_2) \cos[(\omega_c + 2\omega_1)t + \pi]$
(b) Lower second order sideband of ω_1	−2	0	$A_c J_{-2}(X_1) J_0(X_2) \cos[(\omega_c - 2\omega_1)t - \pi]$
(c) Upper first order sideband of ω_2	0	1	$A_c J_0(X_1) J_1(X_2) \cos(\omega_c + \omega_2)t$
(d) Lower first order sideband of ω_2	0	−1	$A_c J_0(X_1) J_{-1}(X_2) \cos(\omega_c - \omega_2)t$

Suppose that $\omega_2 = 2\omega_1$ precisely; that is, the modulation is a fundamental plus a second harmonic component. Under this condition, $\omega_c + 2\omega_1 = \omega_c + \omega_2$ and $\omega_c - 2\omega_1 = \omega_c - \omega_2$. In Table 19-5, sidebands (*a*) and (*c*) can now be combined, as can (*b*) and (*d*). Use of Eq. (19-21) and the identity $\cos(C \pm \pi) = -\cos C$ gives the results shown in Table 19-6.

TABLE 19-6. TERMS OF TABLE 19-5 FOR $\omega_2 = 2\omega_1$

Sideband	Components	Amplitude
(a) Upper sideband at $\omega_c + 2\omega_1$	(a) and (c) of Table 19-5	$-A_c J_2(X_1) J_0(X_2) + A_c J_0(X_1) J_1(X_2)$
(b) Lower sideband at $\omega_c - 2\omega_1$	(b) and (d) of Table 19-5	$-A_c J_2(X_1) J_0(X_2) - A_c J_0(X_1) J_1(X_2)$

Now, let X_1 and X_2 be such that $A_c J_2(X_1) J_0(X_2) = A_c J_0(X_1) J_1(X_2)$. For this case, the upper sideband at $\omega_c + 2\omega_1$ vanishes, but the lower sideband does not. This case illustrates that the spectrum of an FM wave can be nonsymmetrical with respect to the carrier. This cannot happen in amplitude modulation. The result implies that suppression of one sideband of an FM signal may result in distortion; that is, the original modulation function information is not always contained equally in both of the sidebands and some of this information may be lost if only one sideband is used. As a numerical example of this special case, let $X_1 = 1$. From Table 19-2, it follows that for the required condition, $0.115\, A_c J_0(X_2) = 0.765\, A_c J_1(X_2)$. Using tables, it is found that $X_2 = 0.298$, $J_0(X_2) = 0.979$, and $J_1(X_2) = 0.147$. If $M(t)$ is a PM signal, Eq. (19-12) can be used to show that the amplitude of the second harmonic must be 10.5 db below the fundamental. Similarly, if $M(t)$ is an FM signal, the amplitude of the second harmonic must be 4.5 db below the fundamental [from Eq. (19-13)].

Phase and Frequency Modulation by Three or More Tones

Some very useful relations, applicable to the modulation produced by random noise, for example, can be obtained if the general case where the modulation consists of N sinusoids is considered. To fully illustrate some of the mathematical manipulations involved, it is convenient to analyze first one additional specific case—namely, the case where the modulating signal consists of three sinusoids. For this case the modulated carrier can be written as

$$M(t) = A_c \cos\,(\omega_c t + X_1 \cos \omega_1 t + X_2 \cos \omega_2 t + X_3 \cos \omega_3 t) \qquad (19\text{-}32)$$

As in the previous analysis, this equation will be written as

$$M(t) = A_c \cos\left[\left(\frac{\omega_c t}{3} + X_1 \cos \omega_1 t\right) + \left(\frac{\omega_c t}{3} + X_2 \cos \omega_2 t\right) + \left(\frac{\omega_c t}{3} + X_3 \cos \omega_3 t\right)\right] \qquad (19\text{-}33)$$

To simplify notation for the next steps, let

$$\left.\begin{aligned} \eta_1 &= \frac{\omega_c t}{3} + X_1 \cos \omega_1 t \\ \eta_2 &= \frac{\omega_c t}{3} + X_2 \cos \omega_2 t \\ \eta_3 &= \frac{\omega_c t}{3} + X_3 \cos \omega_3 t \end{aligned}\right\} \qquad (19\text{-}34)$$

Substituting Eq. (19-34) into Eq. (19-33) gives

$$M(t) = A_c \cos\,(\eta_1 + \eta_2 + \eta_3) \qquad (19\text{-}35)$$

Equation (19-25) is now applied to expand Eq. (19-35). Thus,

$$M(t) = A_c\,[\cos \eta_1 \cos\,(\eta_2 + \eta_3) - \sin \eta_1 \sin\,(\eta_2 + \eta_3)] \qquad (19\text{-}36)$$

This equation can be further expanded by using Eq. (19-25) and the identity

$$\sin\,(A + B) = \sin A \cos B + \cos A \sin B \qquad (19\text{-}37)$$

Thus, Eq. (19-36) becomes

$$\begin{aligned} M(t) = A_c\,(&\cos \eta_1 \cos \eta_2 \cos \eta_3 - \cos \eta_1 \sin \eta_2 \sin \eta_3 \\ &- \sin \eta_1 \sin \eta_2 \cos \eta_3 - \sin \eta_1 \cos \eta_2 \sin \eta_3) \end{aligned} \qquad (19\text{-}38)$$

Note that to write $M(t)$ in the form of Eq. (19-38), the double angle formulas of the forms given by Eqs. (19-25) and (19-37) have been applied twice. For the general case of N modulating sinusoids, these formulas must be applied $N-1$ times to reduce the equation for $M(t)$ to the same form as Eq. (19-38).

Each cosine and sine term in Eq. (19-38) can be expanded by the Bessel function identities given in Eqs. (19-17) and (19-18). For example, using Eq. (19-18),

$$\begin{aligned} \cos \eta_1 &= \cos\left(\frac{\omega_c t}{3} + X_1 \cos \omega_1 t\right) \\ &= \sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \cos\left(\frac{\omega_c t}{3} + n_1 \omega_1 t + \frac{n_1 \pi}{2}\right) \\ &= \sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \cos \alpha_1 \end{aligned} \qquad (19\text{-}39)$$

where

$$\alpha_1 = \frac{\omega_c t}{3} + n_1 \omega_1 t + \frac{n_1 \pi}{2} \qquad (19\text{-}40)$$

Similar series expansions can be written for all of the other cosine and sine terms of Eq. (19-38) and would result in the following additional expressions, introduced to condense subsequent equations:

$$\alpha_2 = \frac{\omega_c t}{3} + n_2 \omega_2 t + \frac{n_2 \pi}{2} \qquad (19\text{-}41)$$

$$\alpha_3 = \frac{\omega_c t}{3} + n_3 \omega_3 t + \frac{n_3 \pi}{2} \qquad (19\text{-}42)$$

In the above expressions, n_r is an integer associated with each term and which assumes all values from $n_r = -\infty$ to $n_r = +\infty$. In the three-tone case, r takes the values 1, 2, and 3. In the case of N modulating tones, r assumes integral values from 1 to N.

Having expanded each term of Eq. (19-38) by the appropriate Bessel function and defined α_r, Eq. (19-38) can be written as

$$M(t) = A_c \Bigg[\sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \cos \alpha_1 \sum_{n_2=-\infty}^{\infty} J_{n_2}(X_2) \cos \alpha_2 \cdot \sum_{n_3=-\infty}^{\infty} J_{n_3}(X_3) \cos \alpha_3$$

$$- \sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \cos \alpha_1 \sum_{n_2=-\infty}^{\infty} J_{n_2}(X_2) \sin \alpha_2 \cdot \sum_{n_3=-\infty}^{\infty} J_{n_3}(X_3) \sin \alpha_3$$

$$- \sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \sin \alpha_1 \sum_{n_2=-\infty}^{\infty} J_{n_2}(X_2) \sin \alpha_2 \cdot \sum_{n_3=-\infty}^{\infty} J_{n_3}(X_3) \cos \alpha_3$$

$$- \sum_{n_1=-\infty}^{\infty} J_{n_1}(X_1) \sin \alpha_1 \sum_{n_2=-\infty}^{\infty} J_{n_2}(X_2) \cos \alpha_2 \cdot \sum_{n_3=-\infty}^{\infty} J_{n_3}(X_3) \sin \alpha_3 \Bigg] \quad (19\text{-}43)$$

The following relationship may now be used:

$$\sum_{n=-\infty}^{\infty} a_n \sum_{m=-\infty}^{\infty} b_m = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n b_m \quad (19\text{-}44)$$

Applying this relationship to the first term, for example, of Eq. (19-43) permits writing the term in the form

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} J_{n_1}(X_1) J_{n_2}(X_2) \cdot J_{n_3}(X_3) \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \quad (19\text{-}45)$$

If the other three terms of Eq. (19-43) are put into the same form, and if the following definition is made,

$$\beta = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} J_{n_1}(X_1) J_{n_2}(X_2) J_{n_3}(X_3) \quad (19\text{-}46)$$

Eq. (19-43) can then be reduced to the form

$$M(t) = A_c \beta \, (\cos \alpha_1 \cos \alpha_2 \cos \alpha_3 - \cos \alpha_1 \sin \alpha_2 \sin \alpha_3 - \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 - \sin \alpha_1 \cos \alpha_2 \sin \alpha_3) \quad (19\text{-}47)$$

Note that Eq. (19-47) has the same form as Eq. (19-38) and thus can be worked back to the form of Eq. (19-32); thus,

$$M(t) = A_c \beta \cos (\alpha_1 + \alpha_2 + \alpha_3) \quad (19\text{-}48)$$

Substituting Eqs. (19-40), (19-41), (19-42), and (19-46) into Eq. (19-48) gives the final result:

$$M(t) = A_c \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} J_{n_1}(X_1) J_{n_2}(X_2) J_{n_3}(X_3) \cdot \cos \left[(\omega_c + n_1\omega_1 + n_2\omega_2 + n_3\omega_3) t + (n_1 + n_2 + n_3) \frac{\pi}{2} \right] \quad (19\text{-}49)$$

Note that Eq. (19-49) is of the same form as that obtained in the previous section for the two-tone case [Eq. (19-29)].

The same method of analysis can be applied to the case of N modulating sinusoids. When this is done, the following general result is obtained:

$$M(t) = A_c \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \prod_{r=1}^{N} J_{n_r}(X_r)$$
$$\cdot \cos\left(\omega_c t + \sum_{r=1}^{N} n_r \omega_r t + \sum_{r=1}^{N} n_r \frac{\pi}{2}\right) \quad (19\text{-}50)$$

In this equation, the symbol Π denotes that all N of the J_{n_r} coefficients are multiplied together.

It is apparent from Eq. (19-50) that the determination of the spectral components of an FM or PM wave is a very formidable task for even relatively small values of N. Fortunately, in many practical cases of interest, the index of modulation is sufficiently low so that the amplitudes of the various components can be obtained from approximate expressions derived from Eq. (19-50). These expressions are obtained by applying the series expansions of the Bessel functions to the amplitude coefficient of Eq. (19-50) for each basic spectral component.

As an illustration, consider the amplitude of the component that falls at the carrier frequency. This is obtained from Eq. (19-50) by setting all the $n_r = 0$. Thus, if A_0 denotes the amplitude of the carrier component,

$$A_0 = A_c \prod_{r=1}^{N} J_0(X_r) \quad (19\text{-}51)$$

The series expansion for $J_0(X)$ given in Table 19-3 is now substituted into Eq. (19-51). Multiplying out, collecting terms, and neglecting all powers greater than fourth, yields

$$A_0 = A_c\left(1 - \frac{1}{4}\sum_{r=1}^{N} X_r^2 + \frac{1}{16}\sum_{r=1}^{(N-1)}\sum_{s=r+1}^{N} X_r^2 X_s^2 + \frac{1}{64}\sum_{r=1}^{N} X_r^4\right) \quad (19\text{-}52)$$

Equation (19-52) can be written more compactly if an additional small approximation is made. This approximation is to double the coefficient of the last term, thereby introducing an error equal to $1/64 \sum_{r=1}^{N} X_r^4$. However, in so doing, the last two terms of the equation may be combined to reduce Eq. (19-52) to

$$A_0 = A_c\left[1 - \frac{1}{4}\sum_{r=1}^{N} X_r^2 + \frac{1}{32}\left(\sum_{r=1}^{N} X_r^2\right)^2\right] \quad (19\text{-}53)$$

The series expansion for ϵ^{-a} is

$$\epsilon^{-a} = 1 - a + \frac{a^2}{2} - \cdots \quad (19\text{-}54)$$

It should now be observed that Eq. (19-53) has the same form as this series expansion. If the parameter D_ϕ is defined as

$$D_\phi = \frac{1}{2}\sum_{r=1}^{N} X_r^2 \quad (19\text{-}55)$$

it follows that Eq. (19-53) can be expressed as

$$A_0 = A_c\left(1 - \frac{D_\phi}{2} + \frac{D_\phi^2}{8}\right)$$
$$= A_c\,\epsilon^{-D_\phi/2} \quad (19\text{-}56)$$

Equation (19-56) is the approximation desired. It is valid provided that the peak phase deviation X_r of each of the modulating signal components is small, thereby permitting the approximations used to obtain Eqs. (19-52) and (19-53) to be made. More specifically, it is required that $\sum_{r=1}^{N} X_r^2$ be less than unity. The parameter D_ϕ has an important significance. Since it is assumed that each modulating signal component is sinusoidal, the rms phase deviation produced by each component is equal to $X_r/\sqrt{2}$, or the mean square deviation is $X_r^2/2$. Hence, D_Φ is the mean-square phase deviation of the total modulating signal, and $\sqrt{D_\Phi}$ equals the rms phase deviation resulting from the total signal.

The techniques used to obtain Eq. (19-56) can be applied to find approximate expressions for the amplitudes of the other sideband components. Table 19-7 tabulates the results that can be obtained.

The utility of the approximations tabulated in Table 19-7 arises from the fact that numerical methods can be used to calculate the power spectrum of the FM wave for any baseband signal provided

TABLE 19-7. APPROXIMATE EXPRESSIONS FOR AMPLITUDES OF SPECTRAL COMPONENTS (RELATIVE TO CARRIER) FOR N-TONE MODULATION

Component	Frequency	Amplitude and relative phase
Zero order (carrier)	ω_c	$\epsilon^{-D_\Phi/2}$
First order	$\omega_c \pm \omega_r$	$\pm \frac{1}{2} X_r \epsilon^{-D_\Phi/2}$
Second order	$\omega_c \pm 2\omega_r$	$\frac{1}{8} X_r^2 \epsilon^{-D_\Phi/2}$
Second order	$\omega_c \pm \omega_r \pm \omega_s$	$\pm \frac{1}{4} X_r X_s \epsilon^{-D_\Phi/2}$
Third order	$\omega_c \pm 3\omega_r$	$\pm \frac{1}{48} X_r^3 \epsilon^{-D_\Phi/2}$
Third order	$\omega_c \pm 2\omega_r \pm \omega_s$	$\pm \frac{1}{16} X_r^2 X_s \epsilon^{-D_\Phi/2}$
Third order	$\omega_c \pm \omega_r \pm \omega_s \pm \omega_t$	$\pm \frac{1}{8} X_r X_s X_t \epsilon^{-D_\Phi/2}$

$$D_\Phi = \frac{1}{2} \sum_{r=1}^{N} X_r^2 \leq \frac{1}{2}$$

Sign of amplitude term is determined by giving X_r, X_s, and X_t the same signs as ω_r, ω_s, and ω_t.

that (1) the baseband signal can be expressed as a finite summation of N sinusoids, and (2) the mean-square phase deviation, D_Φ, is no greater than 0.5. Such a baseband signal is multichannel telephony, if the assumption is made that each talker can be represented by a single sinusoid. In making the calculations, a power summation is made of all the products falling at each sideband frequency. The product count results given in Chap. 10 are directly applicable.

Phase Modulation by a Band of Random Noise

When the baseband signal consists of many single-sideband, frequency-multiplexed telephone channels, it is often convenient, both for analysis purposes as well as system testing, to simulate the baseband signal by an equivalent band of random noise. As might be suspected, the determination of the sideband spectrum when the modulating signal consists of random noise involves considerable analysis, and the reader is referred to Reference 4, for example, for a detailed illustration of the problem. A particular case of interest, often assumed in the analysis of a radio system, is that of pure phase modulation by a band of random noise extending uniformly across the baseband from 0 to f_b cps. For this case, the power density of the sideband spectrum can be shown to be

$$s(f) = \frac{\epsilon^{-D_\Phi}}{2f_b} \left\{ D_\Phi \left(\frac{1-x}{2}\right)^0 + \frac{D_\Phi^2}{2!} \left(\frac{2-x}{2}\right)^1 + \frac{D_\Phi^3}{3!2!} \left[\left(\frac{3-x}{2}\right)^2 - 3\left(\frac{1-x}{2}\right)^2 \right] + \frac{D_\Phi^4}{4!3!} \left[\left(\frac{4-x}{2}\right)^3 - 4\left(\frac{2-x}{2}\right)^3 \right] + \cdots \right\} \quad (19\text{-}57)$$

where

$s(f)$ = power spectral density (watts) per cycle of bandwidth.

D_Φ = mean-square phase deviation of the total noise signal (radians squared).

f_b = top baseband frequency (cps).

$x = \frac{|f_c - f|}{f_b}$, where f_c is the carrier frequency.

( ) indicates that the enclosed term goes to zero when the quantity $n - x$ is negative.

In this equation, the factor ϵ^{-D_Φ} represents the power in the modu-ıted carrier relative to an unmodulated carrier of unity amplitude. Iote that this is the same result as tabulated in Table 19-7 for the ero order (carrier) component, since the power is proportional to he square of the amplitude and $(\epsilon^{-D_\Phi/2})^2 = \epsilon^{-D_\Phi}$. Figure 19-6 shows he quantity $s(f)2f_b$, expressed in db with respect to the unmodulated arrier, for several values of the root-mean-square phase deviation,

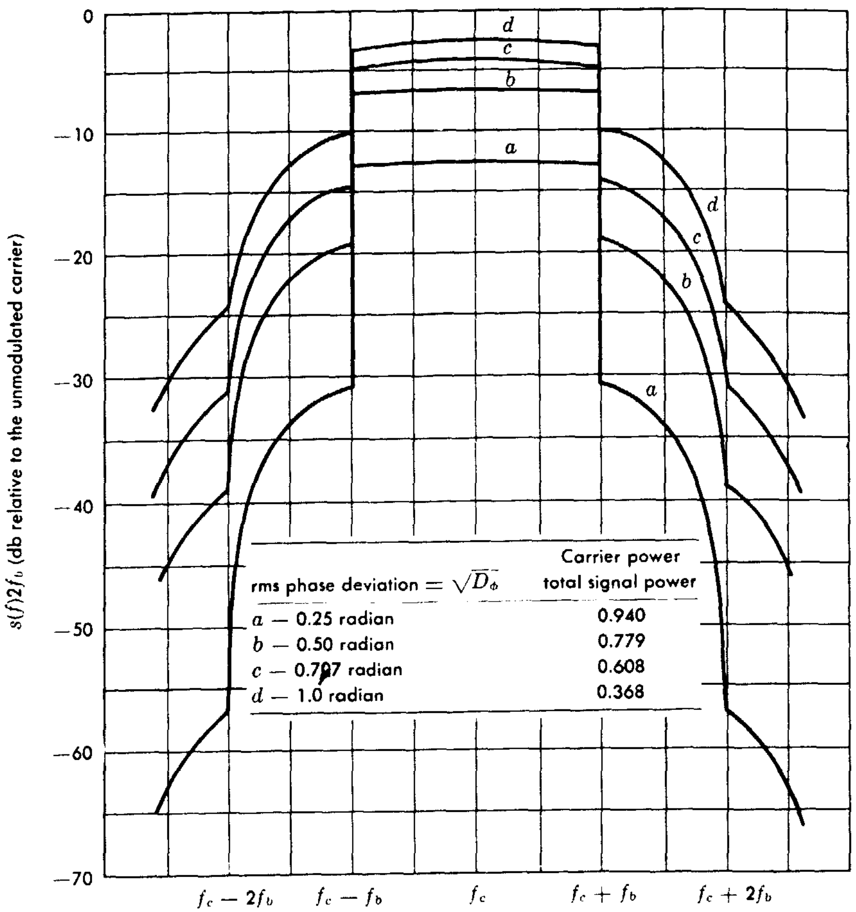


FIG. 19-6. Sideband spectra of carrier, phase-modulated by a baseband signal consisting of a flat band of random noise which extends from 0 to f_b cps.

$\sqrt{D_\Phi}$. Although a curve is shown for $\sqrt{D_\Phi} = 1$, approximations made in the analysis necessitate that $\sqrt{D_\Phi} \leqq 0.707$ (i.e., $D_\Phi^2 \leqq 1/2$) for greatest accuracy. This is the same restriction imposed on the approximations made in deriving the coefficients for Table 19-7.

A qualitative understanding of the shape of the spectra in Fig. 19-6 can be obtained from the following considerations. First order sideband components are formed by the modulation of the carrier and the individual components of the baseband or modulating signal. These sideband components will fall within the band bounded by $f_c \pm f_b$. Since the spectrum of the modulating signal has been assumed flat, the spectrum of the first order sideband components will also be flat versus frequency. This is true even though the amplitude of each sideband component will, of course, be a function of the amplitude of all the other components, as previously discussed in connection with Eq. (19-29).

Second order sideband components, which fall within the band bounded by $f_c \pm 2f_b$, arise from combinations involving the carrier frequency and any second order combination of baseband frequencies such as $A + B$, $A - B$, or $2A$. The number of products formed is greatest in the vicinity of the carrier, with the result that the power in the second order sidebands is maximum around f_c and drops off to zero at frequencies greater than $f_c \pm 2f_b$.

In a similar manner, third order sideband components, which fall in the region bounded by $f_c \pm 3f_b$, arise from combinations of the carrier with third order combinations of baseband frequencies. Again, more products are formed near the carrier frequency, so that the power in the third order sidebands has a broad maximum in the $f_c \pm f_b$ portion of the spectrum and drops to zero at $f_c \pm 3f_b$.

The result of power addition of the higher order components to the first order sidebands accounts for the curvature in the spectrum between $f_c - f_b$ and $f_c + f_b$ in Fig. 19-6. Notice that this curvature increases in going from a low-phase deviation (curve a) to a high deviation (curve d). This is because the power in the second and third order sidebands builds up relatively rapidly as the phase deviation increases. This is analogous to the way second and third order modulation products increase relative to the fundamental as the input to a nonlinear device is increased. The same effect accounts for the relatively slow falloff of higher order sidebands shown by curve d, as against the rapid falloff of curve a.

Spectra for High-Modulation Index

In much of the preceding analysis, approximations have been made which have restricted the results to low-index systems, where the mean-square phase deviation is one-half or less. An alternative technique of approximating the spectrum of the FM wave, applicable to sufficiently high-index systems, is generally known as the quasi-stationary method of analysis. The basic idea may be illustrated by considering an FM wave in which the carrier is 70 mc and the modulating signal is a 100-cps square wave which deviates the carrier ± 1 mc. Then half of the time the FM wave is at 71 mc and half of the time at 69 mc. Thus, a spectrum analyzer would show the spectrum as two spikes, or concentrations of power, at 69 mc and at 71 mc. Each would carry half the total power, or have 0.707 the amplitude, of the unmodulated carrier. If the modulating signal is triangular, so that the frequency sweeps linearly back and forth between 69 mc and 71 mc, the spectrum is clearly essentially continuous and of uniform amplitude between 69 mc and 71 mc. If the effective bandwidth of the spectrum analyzer is, say, 20 kc, the FM wave will be in its passband 1 per cent of the time. The analyzer will then show the uniform amplitude as 10 per cent of that of the unmodulated carrier.

Although the quasi-stationary approach often gives useful results, it should always be viewed with suspicion and used with caution. For low-index systems, it gives wrong results and should be rejected outright. For example, suppose the 70-mc carrier is to be deviated ± 100 kc by a 1-mc square wave. Then the quasi-stationary method says that half the power is at 69.9 mc and half at 70.1 mc. This is completely wrong. There is no power at those frequencies. A correct analysis, using Eq. (19-50) or the approximations of Table 19-7, shows the spectrum to have components spaced at 1-mc intervals around 70 mc. In fact, to pass a 1-mc square wave reasonably well, spectrum components out to ± 10 mc from the carrier would need to be transmitted.

The quasi-stationary approach may often be used when the index is greater than 10, and the low-index approach of Table 19-7 may be used when the mean-square deviation, D_Φ, is less than one-half. This leaves an area of medium indices where no suitable approximation is at present known, and for which any analytical approach becomes quite difficult. One technique is to bracket the problem, by examining a low-index case and one of high-index, and arguing that the medium-index case falls between.

Exponential Notation and Vector Representation

In many instances the use of exponential notation for periodic functions has advantages over the trigonometric notation which has been used thus far in this chapter. A particularly useful application is in the vector representation of AM and PM waves as an aid in understanding the respective modulation processes. This will be considered here.

A sinusoidal carrier $\cos \omega_c t$ can be written as

$$\begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} e^{j\omega_c t}$$

since

$$e^{j\omega_c t} = \cos \omega_c t + j \sin \omega_c t \tag{19-58}$$

The exponential $e^{j\omega_c t}$ is a rotating vector of unit length in the complex plane, and its real part is merely its projection on the real axis. This vector is shown for several values of time in Fig. 19-7.

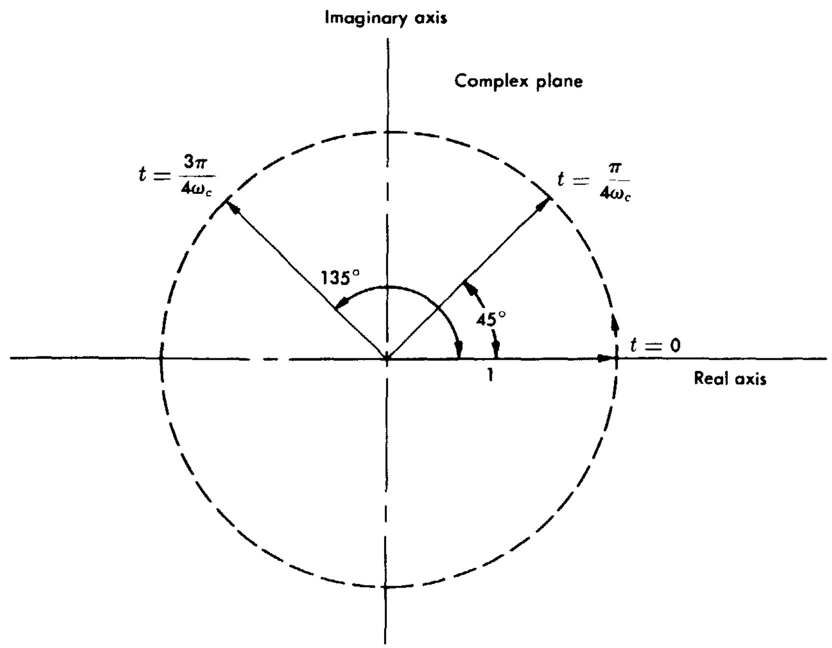


FIG. 19-7. Vector diagram of $e^{j\omega_c t}$ for various times.

An amplitude-modulated wave with 50 per cent modulation (modulation index = 1/2) will now be examined. This can be written as

$$M(t) = \left(1 + \frac{1}{2}\cos \omega_1 t\right)\cos \omega_c t$$
$$= \cos \omega_c t + \frac{1}{4}\cos(\omega_c + \omega_1)t + \frac{1}{4}\cos(\omega_c - \omega_1)t \quad (19\text{-}59)$$

This equation may be written in exponential notation as

$$M(t) = \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix}\left(e^{j\omega_c t} + \frac{1}{4}e^{j(\omega_c+\omega_1)t} + \frac{1}{4}e^{j(\omega_c-\omega_1)t}\right)$$
$$= \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix} e^{j\omega_c t}\left(1 + \frac{1}{4}e^{j\omega_1 t} + \frac{1}{4}e^{-j\omega_1 t}\right) \quad (19\text{-}60)$$

In this form the carrier vector is multipled by the sum of a stationary vector and two rotating vectors of equal size which rotate in opposite directions. As may be seen in Fig. 19-8, the sum of these three vectors is always real and, consequently, acts only to modify the length of the rotating carrier vector. This produces amplitude modulation as expected.

For the purpose of comparison, the frequency-modulated wave of Eq. (19-22) will be similarly represented. If the index of modulation is taken as 1/2, the second and higher order sidebands will be small enough that they may be neglected. The constant multiplier A_c will be considered equal to unity. Thus, for this case,

$$M(t) = \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix}\Big[J_0(1/2)\, e^{j\omega_c t} + J_1(1/2)\, e^{j[(\omega_c+\omega_1)t+\pi/2]}$$
$$+ J_1(1/2)\, e^{j[(\omega_c-\omega_1)t+\pi/2]}\Big] \quad (19\text{-}61)$$

$$M(t) = \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix} e^{j\omega_c t}\Big[J_0(1/2) + J_1(1/2)\, e^{j(\omega_1 t+\pi/2)}$$
$$+ J_1(1/2)\, e^{-j(\omega_1 t-\pi/2)}\Big] \quad (19\text{-}62)$$

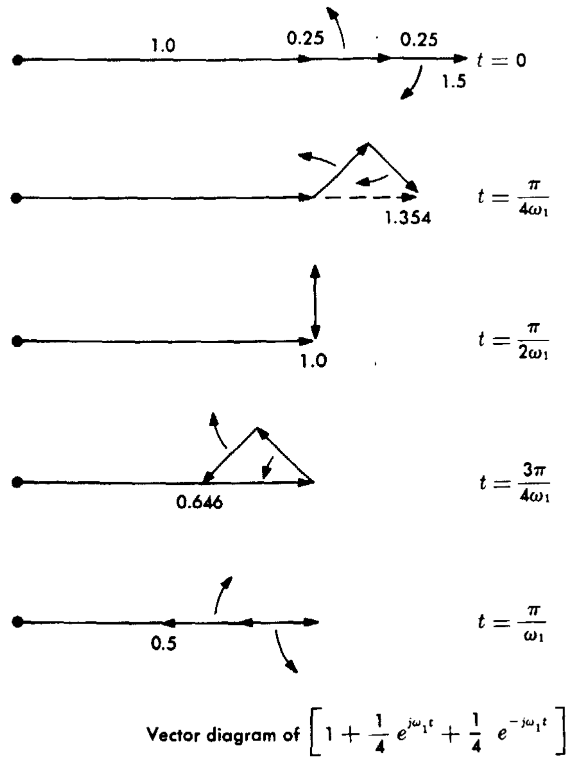


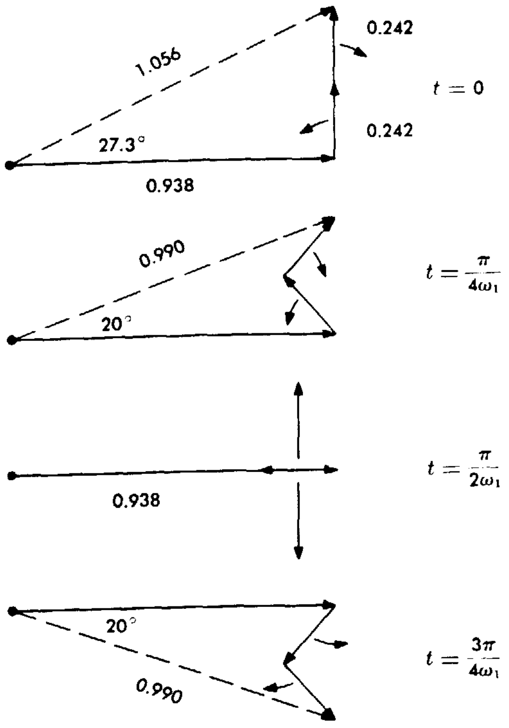
FIG. 19-8. Amplitude modulation — index of modulation = ½.

The multiplying vector, after the constants are evaluated, becomes

$$\left(0.938 + 0.242\, e^{j(\omega_1 t+\pi/2)} + 0.242\, e^{-j(\omega_1 t-\pi/2)}\right)$$

This vector is plotted for several values of time in Fig. 19-9, and it may be seen that it has an essentially constant amplitude but a variable phase. This, then, corresponds to phase modulation of the carrier. If all of the higher order sidebands are retained, the multiplying vector would have a constant amplitude of unity, and the maximum phase deviation would be exactly 28.6°, or 1/2 radian.

Several interesting conclusions may be observed from this comparison. A low-index PM or FM wave and an AM wave are similar in the sense that they both contain the carrier and the same first order



Vector diagram of

$[0.938 + 0.242\, e^{j(\omega_1 t + \pi/2)} + 0.242\, e^{-j(\omega_1 t - \pi/2)}]$

FIG. 19-9. Phase modulation — index of modulation = ½.

sideband frequency components. In fact, for the low-index case, the amplitudes of the first order sidebands are approximately the same for both waves when the indices are equal. The important difference is in the phase of the sideband components. It may be expected, therefore, that in the transmission of an FM or PM wave, the phase characteristic of the transmission path will be extremely important and that certain phase irregularities could easily convert phase-modulation components into amplitude-modulation components. This will be considered in Chap. 21.

Average Power of an FM or PM Wave

The average power of an FM or PM wave is independent of the modulating signal and is equal to the average power of the carrier

when the modulation is zero. Hence, the modulation process takes power from the carrier and distributes it among the many sidebands but does not alter the average power present. This may be demonstrated as follows by assuming a voltage of the form of Eq. (19-2):

$$E(t) = A_c \cos [\omega_c t + \phi(t)] \tag{19-63}$$

The instantaneous power delivered by $E(t)$ to a resistance R becomes

$$\begin{aligned} P(t) &= \frac{E^2(t)}{R} \\ &= \frac{A_c^2}{R} \cos^2 [\omega_c t + \phi(t)] \\ &= \frac{A_c^2}{R}\left\{\frac{1}{2} + \frac{1}{2}\cos [2\omega_c t + 2\phi(t)]\right\} \end{aligned} \tag{19-64}$$

The average power is given by the zero frequency terms in the expression above since non-zero frequency terms have an average value of zero. From the previous analysis of FM and PM waves, the second term in Eq. (19-64) can be assumed to consist of a large number of sinusoidal sideband components about a carrier frequency of $2\omega_c$ radians/second, and, therefore, the average power contributed by each of these terms is zero. Thus, the average power is simply

$$P_{avg} = \frac{A_c^2}{2R} \tag{19-65}$$

This, of course, is the same as the average power in the absence of modulation.

Bandwidth Required for FM Waves

For the low-index case, where the rms phase deviation is less than 0.707 radian, Table 19-7 and Fig. 19-6 show that for a complex signal of many frequencies, most of the signal information is carried by the first order sidebands. It follows that the bandwidth required is at least twice the frequency of the highest frequency component of interest in the modulating signal. This would permit the transmission of the entire first order sideband.

For high-index signals, the quasi-stationary approach indicates that frequencies out to at least the peak frequency deviation need to be transmitted. Detailed examination of the Bessel function coefficients also leads to this conclusion. For the high-index case, then, the minimum bandwidth should be at least twice the peak frequency deviation.

A general rule of thumb (first stated by J. R. Carson in an unpublished memorandum dated August 28, 1939) is that the minimum bandwidth required for the transmission of an FM or PM signal is equal to two times the sum of the peak frequency deviation and the highest modulating frequency to be transmitted. This rule gives results which agree quite well with the bandwidths actually used in the Bell System. It should be realized, however, that this is only an approximate rule and that the actual bandwidth required is to some extent a function of the modulating signal and the quality of transmission desired. A more complete discussion may be found in Reference 5.

EFFECT OF NONLINEARITY ON FM AND PM WAVES

Effect of a Nonlinear Input-Output Characteristic on an FM Wave

Some transmission devices such as electron tubes have nonlinear input-output characteristics which are a source of distortion to an amplitude-modulated signal. This was discussed in Chap. 8. For this purpose, electron tube nonlinearity was expressed by a power series

$$i_p = a_0 + a_1 e_g + a_2 e_g^2 + a_3 e_g^3 \qquad (19\text{-}66)$$

The effect of this same characteristic on an FM signal will now be considered. The FM signal will be taken as

$$e_g = A_c \cos [\omega_c t + \phi(t)] \qquad (19\text{-}67)$$

Substitution in Eq. (19-66) then gives

$$i_p = a_0 + a_1 A_c \cos [\omega_c t + \phi(t)] + a_2 A_c^2 \cos^2 [\omega_c t + \phi(t)] + a_3 A_c^3 \cos^3 [\omega_c t + \phi(t)] \qquad (19\text{-}68)$$

The terms may be expanded and collected:

$$i_p = \left(a_0 + \frac{1}{2} a_2 A_c^2\right) + \left(a_1 A_c + \frac{3}{4} a_3 A_c^3\right) \cos [\omega_c t + \phi(t)] + \frac{1}{2} a_2 A_c^2 \cos [2\omega_c t + 2\phi(t)] + \frac{1}{4} a_3 A_c^3 \cos [3\omega_c t + 3\phi(t)] \qquad (19\text{-}69)$$

The output wave consists of a d-c term and three FM waves centered respectively at the three frequencies ω_c, $2\omega_c$, and $3\omega_c$. Assume for the moment that a filter can be used to extract the FM wave centered at ω_c; the output becomes

$$\text{Filter output} = \left(a_1 A_c + \frac{3}{4} a_3 A_c^3\right) \cos [\omega_c t + \phi(t)] \qquad (19\text{-}70)$$

The nonlinear characteristic has done nothing more than modify the gain. This is an important difference between AM and FM and is one of the reasons why FM is used in the microwave systems where nonlinear operation of amplifiers and other devices has thus far been unavoidable at the desired output levels.

Consider now the restriction made to achieve the desired output above. It is necessary to separate the FM wave centered at ω_c from the one centered at $2\omega_c$. Denote the peak frequency deviation by ΔF and the top baseband frequency by f_b cps; by Carson's rule the FM sidebands of appreciable power centered about the carrier frequency, f_c cps, extend upward to a frequency of $(f_c + \Delta F + f_b)$ cps. Similarly, the sidebands centered about the carrier at $2f_c$ extend downward to a frequency of $(2f_c - 2\Delta F - f_b)$. The $2\Delta F$ is required here because the index of modulation of the FM wave at $2f_c$ is twice as great as the index of modulation of the wave at f_c. Thus, the frequency deviation will also be twice as great. If the two sidebands are not to overlap, the following restriction is obtained:

$$2f_c - 2\Delta F - f_b \geqq f_c + \Delta F + f_b \qquad (19\text{-}71)$$

from which

$$f_c \geqq 3\Delta F + 2f_b \qquad (19\text{-}72)$$

If this restriction is met, it is possible to recover the fundamental FM wave without significant distortion.

Limiters

Some devices which are used in FM systems produce distortion if the frequency-modulated wave is also amplitude-modulated. Traveling-wave tubes and some types of FM demodulators are examples. They make it necessary either to prevent amplitude modulation or to provide methods for suppressing it. Amplitude modulation can be caused by imperfect FM modulators or by transmission deviations which may convert FM sidebands into AM sidebands. It is therefore not easily prevented.

A limiter is a highly nonlinear device which suppresses any incidental amplitude modulation of a carrier with little effect on phase modulation. An *ideal* limiter is one which removes AM completely and has no effect at all on PM or FM. This is strictly a mathematical concept. A *real* limiter reduces AM to a fraction of its original value. In a good design, the AM index may be reduced by a factor of 100; this is frequently referred to as 40 db of limiting (i.e., 20 log 100 = 40 db). Furthermore, in an actual limiter, there is some conversion of the AM at the input to PM at the output. In a good limiter, the PM index will be only a small fraction of the AM index, perhaps 2 per cent; this, too, is often measured in db. In this example, the AM to PM conversion is 20 log 0.02 = −34 db. In contrast, a traveling-wave tube may have −6 db of AM to PM conversion. Finally, a real limiter usually has some bandwidth distortion due to the necessity to extract the FM wave centered at ω_c from the total output, as discussed in the previous section.

Because limiters are highly nonlinear, analysis involving them is difficult. It is also too lengthy and specialized to be included here. However, a brief outline of the present status is of interest.

The analysis may be broken into two separate problems. The first is to examine the limiter as a device in order to predict its external performance (amount of limiting, AM to PM conversion, etc.) in terms of its internal construction. The second takes the external performance as given data, and examines the use of the device in a complete transmission system. In neither of these areas can it be said that the analytical situation is satisfactory.

With respect to the first problem, a limiter characteristic frequently assumed for study is shown in Fig. 19-10. When the input signal is sufficiently large, the peaks of the FM wave are clipped, and most of the undesired AM is removed. The flat tops on the output signal infer the presence of many harmonics, which may be removed with a filter. This input-output characteristic is called a partial limiter characteristic since the amount of AM limiting is dependent on the magnitude of the input signal; the characteristic is idealized because it is discontinuous and because it implies infinite bandwidth. For a truly ideal limiter, the sloping part of the characteristic would be vertical. The output would then be independent of the input amplitude. In fact, the output would retain only the zero-crossing information of the input signal. This, in turn, implies that the information in an FM wave is defined completely by its zero crossings.

As a somewhat better approximation to reality, the circuit shown in Fig. 19-11 has been studied. Here the diodes are idealized; that is, they are open circuit so long as $|V(t)| < E$, but otherwise carry whatever current is necessary to limit $V(t)$ to values $+E$ or $-E$.

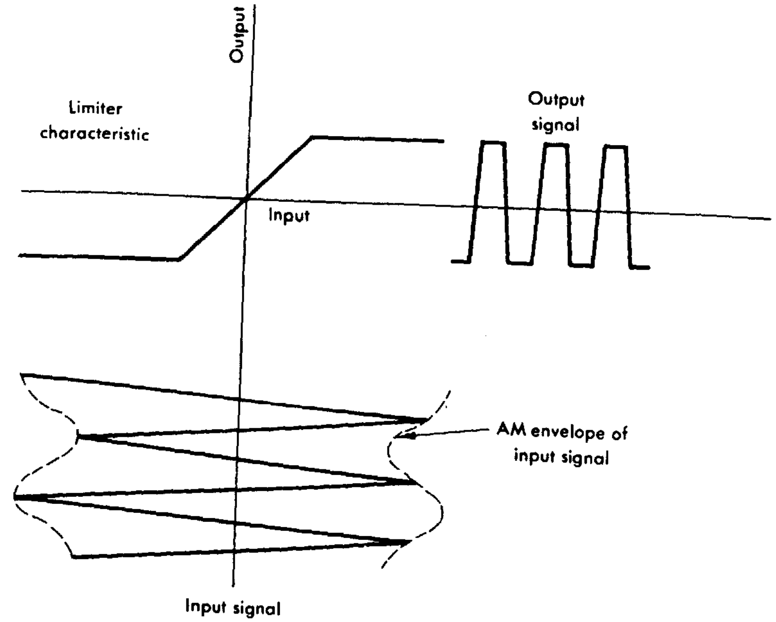


FIG. 19-10. Idealized partial limiter characteristic.

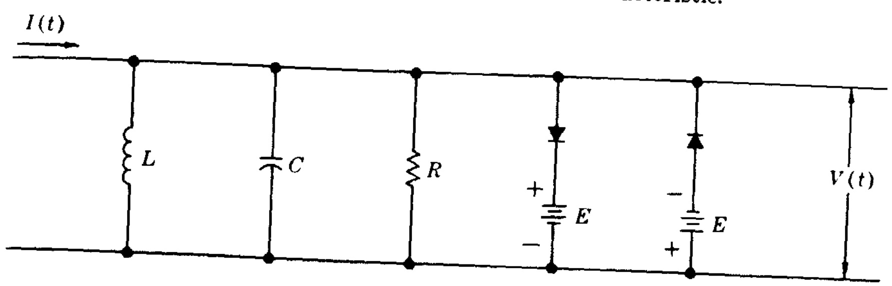


FIG. 19-11. Idealized real limiter.

With L and C omitted, this idealized limiter has a characteristic like Fig. 19-10, but when they are included the situation is drastically changed. With a steady sine-wave input current of the resonant frequency of L and C, the output voltage wave is as shown in Fig. 19-12. This is far from the clipped sine wave usually predicted. Furthermore, the analysis shows that the location of the zero crossings depends on the input amplitude; i.e., this simple limiter has level-to-phase, or AM to PM conversion.

Calculation of the performance with sine-wave inputs at frequencies off resonance (the quasi-stationary approach) gives some

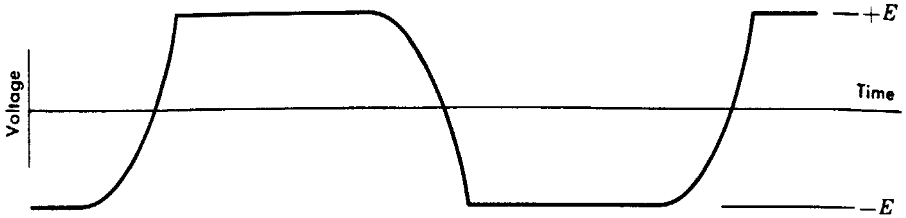


FIG. 19-12. Waveshape of $V(t)$ for limiter of Fig. 19-11.

idea of the limiter action with high-index signals. It appears to be quite complex. The next step would be to use as the input a low-index FM or AM wave comprising only the carrier and two small first order sidebands. This has not been done.

With respect to the second problem, i.e., the performance of a limiter in a complete transmission system, it is necessary to consider simultaneous AM and PM of the carrier. This may be written as

$$M(t) = A_c\left[1 + \sum_i a_i \cos(\omega_i t + \theta_i)\right] \cdot \cos\left[\omega_c t + \sum_j x_j \sin(\omega_j t + \theta_j)\right] \quad (19\text{-}73)$$

As can be imagined, the expansion of this is lengthy, as it involves not only the usual AM and PM sidebands, but also interaction sidebands involving products of the a_i and x_j components. However, some numerical analysis has been carried out for low-index systems, using a digital computer. In these calculations, the essentially continuous baseband spectrum of a multiplex telephone system was simulated by ten uniformly spaced single frequencies. Each repeater section was represented by a transmission network followed by a limiter; there were ten such combinations in tandem. The gain and phase distortions of the networks and the amount of AM limiting and AM to PM conversion in the limiters were given numerical values. Only first and second order distortions were considered.

The results defy a succinct summary. In general, they show that it is theoretically impossible to equalize perfectly the effect of transmission deviations of a multi-repeatered system with tandem limiters by i-f phase and amplitude equalization at the receiving end of the system, and this is borne out by field experience. It is better to reduce the deviations of each repeater by design and by manufacturing control. That which is unavoidable should be equalized at as close intervals as is practical. However, some types of over-all equalization

are definitely useful. Two such types are gain equalization for roll-off (poorer transmission) at high-baseband frequencies and equalization for linear delay distortion. The latter is responsible for much of the cross-modulation noise in broadband microwave systems.

The basic difficulty is that an FM system with limiters is inherently highly nonlinear. Many familiar concepts based on the principle of superposition have to be abandoned. As an illustration of this, consider a transmission phase characteristic which rotates each of the sideband vectors of a low-index FM wave by 45° clockwise. Figure 19-13(a) shows the undistorted wave at $t = 0$; here the carrier amplitude is assumed to be unity and use is made of the low-index approximation to represent the first order sidebands as $X/2$, where X is the modulation index. The vector diagram after trans-

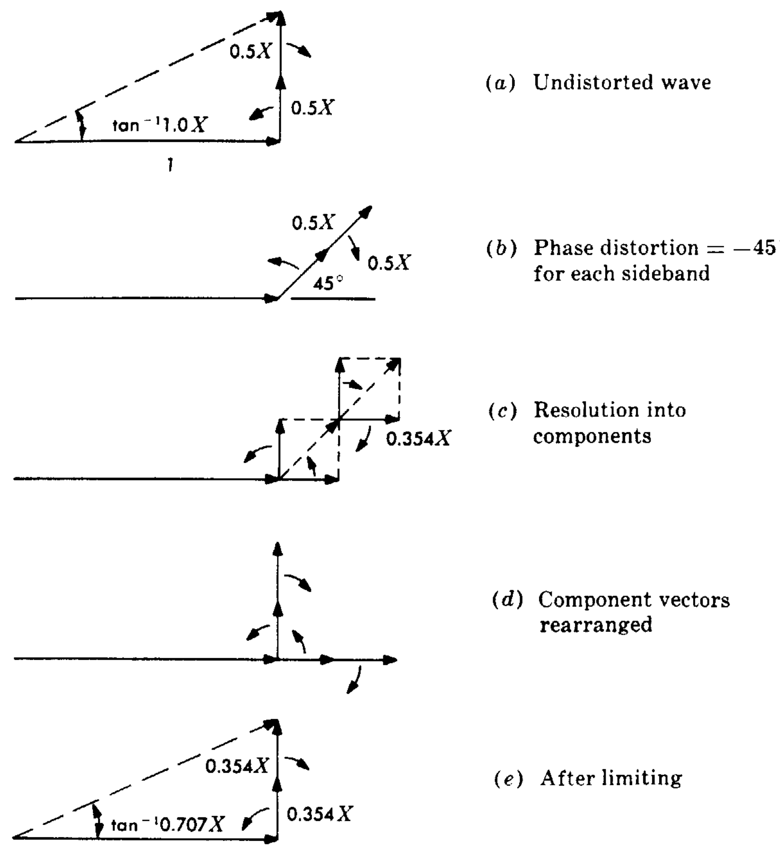


FIG. 19-13. Effects of phase distortion and limiting — phase modulation index $= X$; $t = 0$.

mission distortion is shown in (*b*). Each sideband vector can be resolved into components as shown in (*c*), and rearranged as in (*d*). This is clearly a combination of AM and PM. An ideal limiter removes the AM components, leaving (*e*). This is a pure PM wave, but the index of modulation (i.e., peak phase deviation) has been reduced from X to $0.707X$. The baseband output will thus be reduced 3 db.

Phase distortion has thus produced amplitude distortion at baseband. There is no phase equalization which can restore the sidebands in (*e*) to their original amplitude of $0.5X$; only a gain equalizer can do that. In fact, it is clear that a phase equalizer, placed in the system somewhere between limiters and intended to correct for the original phase distortion (by rotating each sideband vector 45° counterclockwise), will result only in a repetition of the process, so that a second 3-db transmission loss at baseband will ensue.

In general, it can be demonstrated that there is no one-to-one correspondence between the transmission characteristic of the network ahead of the limiter and the necessary equalizer which follows. As a result, measurements for equalization purposes which are made through limiters by ordinary sweep-frequency techniques are not always useful. One partial solution to this problem is the use of a frequency-modulated carrier which is swept slowly across the band of interest. The visual delay and the differential gain and phase measuring sets use this approach.

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Chapter 20

RANDOM NOISE IN FM AND PM SYSTEMS

The unwanted amplitude and phase modulation produced by an interfering sinusoid is analyzed for a low-index FM or PM system. The principles of the analysis are then extended to determine the modulation caused by a flat band of random noise. Numerical examples illustrate the means of applying the results to the computation of noise in a low-index FM or PM system. The problem of analyzing the effect of the interference in a high-index system is briefly discussed. Other topics include the comparison of noise in FM, PM, and AM systems.

From earlier discussions of random noise, it will be recalled that thermal agitation type noise determines a lower limit to the random noise level in any electrical circuit, and that additional noise may be expected from other sources such as electron tubes and transistors. In previous chapters, the emphasis was on random noise in voice-frequency and amplitude-modulation systems. In this chapter, the effect of random noise in phase- and frequency-modulated systems will be considered.

If an unmodulated carrier wave is combined with a band of random noise having constant spectral density, the resultant wave is equivalent to a carrier wave which has been both amplitude- and phase-modulated by random noise. When the resultant wave is demodulated by either an ideal amplitude detector or an ideal phase detector, a random noise output is obtained. Since phase modulation and frequency modulation are so closely related, it is obvious that an FM demodulator will also have a random noise output. However, the output is not the same in an FM system as it is in a PM system. As is demonstrated in this chapter, the noise voltage at the output of a PM system is flat with frequency, whereas the noise voltage at the output of an FM system increases linearly with frequency. This is commonly referred to as the triangular noise spectrum of an FM system.

The analysis in this chapter begins by first considering the unwanted amplitude and phase modulation of a carrier which is produced by an interfering sinusoidal signal, such as a spurious tone in the transmission band. For a sinusoidal interference, the frequency modulation can easily be deduced from the phase modulation. Having developed the necessary equations for this simple case of a single-tone interference (which is, of itself, an important problem in radio systems), the random noise case is treated next by considering the random noise to be the sum of a very large number of equally spaced and randomly phased interfering sinusoids of equal peak amplitudes.

The reader should note that in most of the following sections the carrier to which the interfering sinusoid or noise is added is assumed to be unmodulated. The results, however, are applicable for noise or interfering signals which are added to a low-index FM wave since most of the power is then in the carrier component. The problem of high-index systems is treated briefly in a later section.

Amplitude and Phase Modulation of a Sinusoidal Carrier by an Interfering Sinusoidal Signal

The combination of a sinusoidal carrier and an interfering sinusoidal signal can be written as:

$$M(t) = \underbrace{A_c \cos \omega_c t}_{\text{Carrier}} + \underbrace{A_n \cos\left[(\omega_c + \omega_n)t + \theta_n\right]}_{\text{Interfering sinusoid}} \qquad (20\text{-}1)$$

where

A_c = peak carrier amplitude in volts
ω_c = carrier frequency in radians per second
A_n = peak amplitude of interfering sinusoid in volts
$(\omega_c + \omega_n)$ = frequency of interfering sinusoid in radians per second
θ_n = phase angle of interfering sinusoid

The frequency spectrum of this combined signal is shown in Fig. 20-1. It is not immediately obvious from either the figure or Eq. (20-1) that the combined signal is equivalent to a carrier with frequency ω_c, which has been simultaneously amplitude- and phase-modulated at a radian frequency ω_n. However, it is shown in the

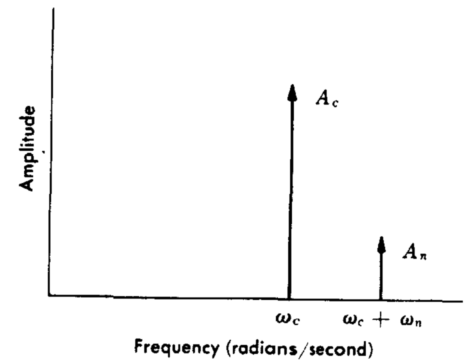

Fig. 20-1. Spectrum — carrier plus interfering sinusoid.

appendix to this chapter that when $A_n << A_c$, an approximation to $M(t)$ can be written as:

$$M(t) \cong A_c\left[1 + \frac{A_n}{A_c}\cos(\omega_n t + \theta_n)\right] \cdot \cos\left[\omega_c t + \frac{A_n}{A_c}\sin(\omega_n t + \theta_n)\right] \qquad (20\text{-}2)$$

This result clearly shows that the carrier has both amplitude and phase modulation at the difference frequency between the interference and the carrier. The peak phase deviation (or index of modulation) in radians is given by the ratio of the amplitude of the interference to the carrier amplitude; that is,

$$\text{Peak phase deviation} = \frac{A_n}{A_c} \text{ radians} \qquad (20\text{-}3)$$

Since the phase modulation is sinusoidal, the rms phase deviation is equal to the peak phase deviation divided by $\sqrt{2}$. Hence,

$$\text{Rms phase deviation} = \frac{A_n}{A_c\sqrt{2}} \text{ radians}$$
$$= \frac{a_n}{A_c} \text{ radians} \qquad (20\text{-}4)$$

where $a_n = A_n/\sqrt{2}$ is the rms amplitude of the interference. The rms phase deviation will be useful later since, for random noise, the rms voltage is more easily defined than the peak voltage.

Another point of importance in the random noise case is that ω_n can be either positive or negative depending on whether the interference frequency is above or below the carrier frequency. Thus, a noise component at a frequency of either $\omega_c + \omega_n$ or $\omega_c - \omega_n$ produces a baseband output at the same frequency ω_n. When two such noise components are simultaneously present (as they usually are), they add on a power basis, since, in general, they can be assumed to arise from uncorrelated voltages.

A Vector Representation of the Single-Tone Interference Case

Equation (20-1) can be written in exponential notation as follows:

$$M(t) = \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix} \left(A_c e^{j\omega_c t} + A_n e^{j[(\omega_c+\omega_n)t+\theta_n]}\right)$$

$$= \begin{bmatrix}\text{Real}\\ \text{part}\\ \text{of}\end{bmatrix} \left[A_c e^{j\omega_c t}\left(1 + \frac{A_n}{A_c} e^{j(\omega_n t+\theta_n)}\right)\right] \qquad (20\text{-}5)$$

The multiplying vector,

$$1 + \frac{A_n}{A_c} e^{j(\omega_n t+\theta_n)},$$

is shown in Fig. 20-2. It is obvious that this vector varies in both amplitude and phase as a function of $\omega_n t$. When $A_n/A_c << 1$, the peak phase deviation is approximately equal to A_n/A_c radians, which is the same result obtained in the previous section.

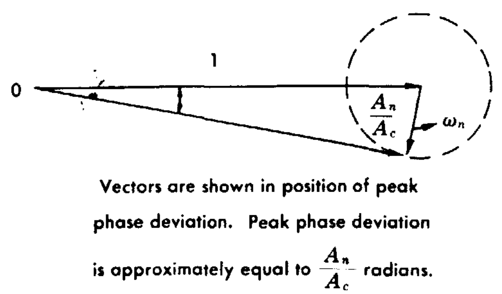

FIG. 20-2. Vector diagram — carrier plus interfering sinusoid.

Frequency Modulation of a Sinusoidal Carrier by an Interfering Sinusoidal Signal

The unwanted frequency modulation produced by the sinusoidal interference is easily obtained from a knowledge of the phase modulation since the instantaneous frequency deviation is defined as the time derivative of the instantaneous phase deviation. When the carrier is much larger than the interference, the instantaneous phase deviation is given in Eq. (20-2) as:

$$\text{Instantaneous phase deviation} = \frac{A_n}{A_c} \sin(\omega_n t + \theta_n) \text{ radians} \qquad (20\text{-}6)$$

Taking the derivative with respect to time gives:

$$\text{Instantaneous frequency deviation} = \frac{A_n\omega_n}{A_c} \cos(\omega_n t + \theta_n) \text{ rad/sec} \qquad (20\text{-}7)$$

The peak frequency deviation is given by:

$$\text{Peak frequency deviation} = \frac{A_n\omega_n}{A_c} \text{ rad/sec}$$

$$= \frac{A_n f_n}{A_c} \text{ cps} \qquad (20\text{-}8)$$

The rms frequency deviation resulting from the interfering sinusoid is given by the peak frequency deviation divided by $\sqrt{2}$. Thus,

$$\text{Rms frequency deviation} = \frac{A_n\omega_n}{A_c\sqrt{2}} \text{ rad/sec}$$

$$= \frac{A_n f_n}{A_c\sqrt{2}} \text{ cps}$$

$$= \frac{a_n f_n}{A_c} \text{ cps} \qquad (20\text{-}9)$$

It should be observed that the peak frequency deviation is a function of the difference frequency $\omega_n = 2\pi f_n$. Consequently, sinusoidal components which are well displaced from the carrier frequency produce larger frequency deviations than sinusoidal components close to the carrier frequency. For the rms phase deviation, it is sufficient to know the ratio of the rms interference voltage, $a_n = A_n/\sqrt{2}$, to the peak carrier voltage, A_c. For the rms frequency deviation, it is necessary to take into account the frequency of the interference.

In Chap. 19 it was pointed out that the index of modulation (or peak phase deviation) of a carrier modulated by a single sinusoid can be expressed as the peak frequency deviation divided by the modulating signal frequency. Comparison of Eqs. (20-3) and (20-8) shows that the same relation applies here for the case of modulation resulting from the presence of an interfering sinusoid. Thus, in general, for sinusoidal modulation, a useful relation to remember is:

$$\frac{\text{Peak phase deviation}}{\text{(or index of modulation)}} = \frac{\text{Peak frequency deviation}}{\text{Modulation frequency}} \quad (20\text{-}10)$$

Illustrative Example 1

Problem

In order to illustrate the concepts described in the preceding sections, an example of a single interfering sinusoid will be considered.

Assume that an FM signal with a 70-mc carrier is frequency-modulated with a 7.75-mc baseband sine wave. Assume also that the peak frequency deviation is 4 mc. A 62-mc sinusoidal interference is added to the FM wave. If the power of the interfering tone is 40 db below that of the FM wave, what is the signal-to-interference ratio at the output of the system?

Solution

Using Eq. (20-10), the index of modulation due to the desired baseband signal is 4/7.75 = 0.52. This represents, therefore, a low-index system with most of the power in the carrier. Little error is incurred by assuming all of the power to be in the carrier. Thus, since the interference power is 40 db below the carrier power, it follows that 20 log $(A_n/A_c) = -40$ db from which $A_n/A_c = 0.01$. The peak phase deviation produced by the interference is, therefore, 0.01 radian. The peak frequency deviation due to the interference is given by Eq. (20-8) and is equal to the product of the peak phase deviation produced by the interference and the difference frequency f_n between the interference and the carrier. Since in this case f_n is 8 mc, the peak frequency deviation is 0.08 mc. This may be compared with the peak frequency deviation due to the transmitted signal to determine the signal to interference ratio, S/N; thus,

$$\frac{\mathrm{S}}{\mathrm{N}} = 20 \log \frac{4}{0.08}$$
$$= 34 \text{ db}$$

Amplitude and Phase Modulation of a Sinusoidal Carrier by a Band of Random Noise

The effects of interference in PM and FM systems due to a band of random noise about the carrier are considered next. Of interest

are (1) the total noise which appears in the baseband (important for television transmission, for example), and (2) the noise in a particular baseband slot (for example, the noisiest channel in a telephone multiplex group).

The method of analysis developed in the preceding sections is directly applicable provided the total noise power in the radio channel is small compared to the carrier power. The random noise can be assumed to consist of an extremely large number of sinusoidal components of incommensurable frequency, of equal amplitude, and of arbitrary phase. It is convenient to analyze the system noise on a per-cycle basis; thus, a band of noise N cycles wide will be thought of as equivalent to N approximately uniformly-spaced sinusoids. Let A_n equal the peak amplitude of a sinusoid having the same power as a band of noise one cycle wide. The combination of the carrier and the band of random noise can be written, therefore, as

$$M(t) = A_c \cos \omega_c t + \sum_{n=1}^{N} A_n \cos [(\omega_c + \omega_n)t + \theta_n] \quad (20\text{-}11)$$

Note that Eq. (20-11) is essentially the same as Eq. (20-1) except that N interfering sinusoids are now being considered instead of only one. The derivation in the appendix to this chapter shows that Eq. (20-11) can be written as

$$M(t) = A_s(t) \cos [\omega_c t + \phi_s(t)] \quad (20\text{-}12)$$

where, when the noise power is much less than the carrier power,

$$A_s(t) \cong A_c \left[1 + \sum_{n=1}^{N} \frac{A_n}{A_c} \cos (\omega_n t + \theta_n) \right] \quad (20\text{-}13)$$

$$\phi_s(t) \cong \sum_{n=1}^{N} \frac{A_n}{A_c} \sin [\omega_n t + \theta_n] \text{ radians} \quad (20\text{-}14)$$

Comparison of these results with Eq. (20-2) shows that, for the conditions assumed, the principle of superposition holds. That is, in the random noise case, the amplitude and phase modulation of the carrier is equal to the summation of the amplitude- and phase-modulation components, respectively, which would have been produced by each noise component separately. In general, the amplitude modulation of the carrier by the random noise can be neglected in FM radio systems if adequate AM suppression (by use of limiters, for example) is provided. The phase modulation, on the other hand, will produce noise in the baseband at the output of the FM or PM detector.

When a single interfering sinusoid was being considered, it was possible to write directly an expression for the peak phase deviation, Eq. (20-3). The peak value of N interfering sinusoids can be defined only if a known relationship exists between the sinusoids, which is not the case here. Thus, for random noise, it is not possible to write an expression for the peak phase deviation which is analogous to Eq. (20-3). The rms phase deviation, however, can be defined. Let the total rms voltage produced by the N sinusoids be a_N. For the case assumed, where $A_1 = A_2 = A_n$, it follows that this rms voltage is

$$\begin{aligned} a_N &= \sqrt{\sum_{n=1}^{N}\left(\frac{A_n}{\sqrt{2}}\right)^2} \\ &= \sqrt{N\left(\frac{A_n}{\sqrt{2}}\right)^2} \\ &= \frac{A_n}{\sqrt{2}}\sqrt{N} \\ &= a_n\sqrt{N} \end{aligned} \qquad (20\text{-}15)$$

where $a_n = A_n/\sqrt{2}$ is the rms amplitude of each sinusoidal component. Using these results, the rms phase deviation of Eq. (20-14) can be written as

$$\begin{aligned} \text{Total rms phase deviation due to band of random noise} &= \frac{a_N}{A_c} \text{ radians} \\ &= \frac{A_n}{A_c\sqrt{2}}\sqrt{N} \text{ radians} \\ &= \frac{a_n}{A_c}\sqrt{N} \text{ radians} \end{aligned} \qquad (20\text{-}16)$$

When $N = 1$, the above expressions reduce to those previously defined for the rms phase deviation due to a single sinusoid; that is,

$$\begin{aligned} \text{Rms phase deviation due to a 1-cycle band of noise} &= \frac{A_n}{A_c\sqrt{2}} \text{ radians} \\ &= \frac{a_n}{A_c} \text{ radians} \end{aligned} \qquad (20\text{-}17)$$

PM System Noise

The preceding analysis can now be applied to the specific problems of noise in the baseband of both PM and FM systems. It will be assumed that the random noise is flat versus frequency over the band from $f_c - f_1$ to $f_c + f_1$, as shown in Fig. 20-3(a), and that the carrier

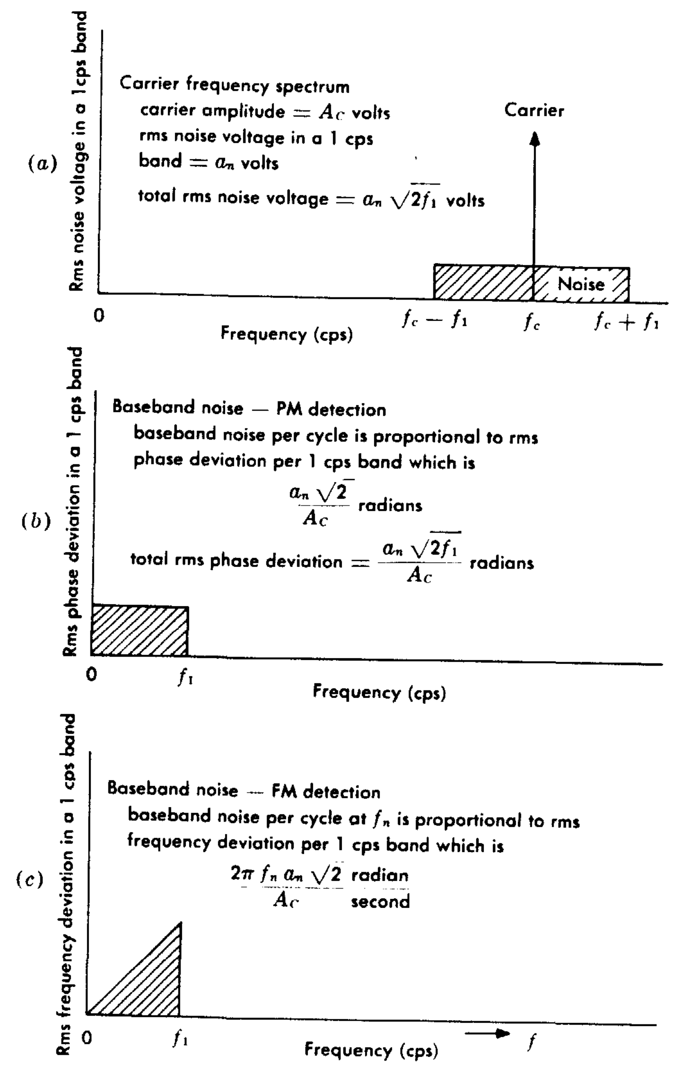

FIG. 20-3. Addition of a flat band of noise to a carrier and resultant baseband noise in PM and FM systems.

power is much greater than the noise power. The rms phase deviation due to the noise in a 1-cycle band at $f_c + f_n$, where $f_n \leqq f_1$, is a_n/A_c radians, as shown in Eq. (20-17). This phase modulation will cause an rms noise voltage to appear at the output of a PM detector in a 1-cycle band centered at baseband frequency f_n. A second noise voltage of equal magnitude will also appear at f_n due to the noise in a 1-cycle band centered at $f_c - f_n$. Since the two noise voltages are uncorrelated, they may be assumed to add on a power basis. Thus, the total rms noise voltage in a 1-cycle band centered at frequency f_n will be proportional to $a_n\sqrt{2}/A_c$. The ratio $a_n\sqrt{2}/A_c$ is, of course, the rms phase deviation produced by two 1-cycle bands of noise [Eq. (20-16) for the case $N = 2$], where the specific interpretation has been made that one band is f_n cycles above the carrier and the other is f_n cycles below the carrier. Thus, for a PM system, at the output of a radio receiver with a phase modulation detector,

$$\text{Rms noise voltage in a 1-cycle band centered at baseband frequency } f_n = C_{\text{PM}} \frac{a_n\sqrt{2}}{A_c} \tag{20-18}$$

where

C_{PM} = transfer constant of PM detector, in output volts per radian phase deviation

and

$\dfrac{a_n\sqrt{2}}{A_c}$ = rms phase deviation, in radians, due to two 1-cycle bands of noise, one at $f_c + f_n$ and the other at $f_c - f_n$, where f_c is the carrier frequency

It is evident from the above analysis that the baseband noise voltage per cycle in a PM system due to flat random noise is independent of f_n. Hence, the noise spectrum at the output of a PM detector is flat versus frequency. The total baseband noise voltage is equal to the equivalent voltage resulting from the power summation of all the 1-cycle band noise voltages. If the baseband extends from 0 to f_1 cps, the total rms noise voltage at the output of the PM detector will be the power summation of f_1 voltages, each of which has an rms amplitude given by Eq. (20-18). Thus,

$$\begin{aligned}\text{Total rms noise voltage in PM system baseband} &= \sqrt{f_1\left(C_{\text{PM}} \frac{a_n\sqrt{2}}{A_c}\right)^2} \\ &= C_{\text{PM}} \frac{a_n}{A_c} \sqrt{2f_1} \text{ volts}\end{aligned} \tag{20-19}$$

where

$\dfrac{a_n}{A_c}\sqrt{2f_1}$ = total rms phase deviation, in radians, due to a band of noise extending from $f_c - f_1$ to $f_c + f_1$

Note that this is the same as Eq. (20-16) for the case $N = 2f_1$.

Figure 20-3(*b*) illustrates the baseband noise spectrum for a PM system. If telephone multiplex were transmitted over such a system, all channels would be equally noisy. To find the noise in dbrn at 0 TL, some relationship between the rms phase deviation in the radio system and the power at the zero transmission level is needed. This relationship could be stated in a number of ways; a similar problem occurs in the FM case and is considered in more detail in the following sections.

FM System Noise

As in the case of the PM system noise, it will be assumed that the random noise is flat versus frequency over the band from $f_c - f_1$ to $f_c + f_1$, as shown in Fig. 20-3(*a*), and that the carrier power is much greater than the noise power. The baseband noise voltage in an FM system is proportional to the rms frequency deviation rather than the rms phase deviation of the carrier. The instantaneous frequency deviation caused by the noise can be obtained by differentiating the expression for the instantaneous phase deviation, Eq. (20-14). The result would be a summation of N cosinusoidal terms, each having a peak amplitude of $\omega_n A_n/A_c$ or an rms amplitude of $\omega_n a_n/A_c$ volts. These amplitudes are, of course, the same as those for the single-tone interference case, Eqs. (20-8) and (20-9), and represent the peak and rms frequency deviations, respectively, caused by the noise at frequency $f_c + f_n$ or $f_c - f_n$. The total rms noise voltage in a 1-cycle band centered at f_n at the output of an FM detector will be the power summation of the rms noise voltages due to the noise at both of these frequencies. Thus, for an FM system, at the output of an FM detector,

$$\text{Rms noise voltage in a 1-cycle band centered at baseband frequency } f_n = C_{\text{FM}} \frac{\omega_n a_n\sqrt{2}}{A_c} \tag{20-20}$$

where

C_{FM} = transfer constant (or deviation sensitivity) of FM detector, in output volts per radian per second frequency deviation and

$$\frac{\omega_n a_n \sqrt{2}}{A_c} = \text{rms frequency deviation, in rad/sec, due to two 1-cycle bands of noise, one at } f_c + f_n \text{ and the other at } f_c - f_n, \text{ where } f_c \text{ is the carrier frequency in cps and } f_n = \omega_n/2\pi$$

Note that the baseband noise voltage per cycle in an FM system varies directly with ω_n and therefore increases linearly with baseband frequency. This is the so-called triangular noise spectrum of an FM system and is illustrated in Fig. 20-3 (*c*).

The total rms noise voltage at the output of the FM detector will be directly proportional to the total rms frequency deviation of the carrier. The total rms frequency deviation may be obtained by integrating the mean square frequency deviation produced by each 1-cycle band of noise and then taking the square root. This is, of course, analogous to a power summation of the components. Thus, if the baseband extends from 0 to f_1 cycles, the total rms noise voltage will be proportional to

$$\text{Total rms frequency deviation of carrier} = \sqrt{\int_0^{f_1} \left(\frac{2\pi f a_n \sqrt{2}}{A_c}\right)^2 df}$$

$$= \frac{2\pi a_n \sqrt{2}}{A_c} \sqrt{\frac{f_1^3}{3}} \text{ rad/sec}$$

$$= \frac{a_n \sqrt{2}}{A_c} \sqrt{\frac{f_1^3}{3}} \text{ cps} \qquad (20\text{-}21)$$

The total rms frequency deviation for a portion of the spectrum is found by changing the limits of integration. A case of particular interest is that in which the bandwidth under consideration is small compared to its separation from the carrier. For example, the top channel in a telephone multiplex signal might occupy a 3-kc band at a baseband frequency of around 4 mc in a typical radio system. Let the bandwidth at the top baseband frequency, f_1, be equal to δf, where $\delta f << f_1$. For this case, the rms frequency deviation will be approximately equal to the power summation of δf components each having

an amplitude given by Eq. (20-20). Thus,

$$\text{Rms frequency deviation due to two } \delta f \text{ bands of noise, one at } f_c + \omega_1/2\pi \text{ and the other at } f_c - \omega_1/2\pi \cong \sqrt{\delta f \left(\frac{\omega_1 a_n \sqrt{2}}{A_c}\right)^2}$$

$$\cong \frac{\omega_1 a_n \sqrt{2\delta f}}{A_c} \text{ rad/sec}$$

$$\cong \frac{a_n f_1 \sqrt{2\delta f}}{A_c} \text{ cps} \qquad (20\text{-}22)$$

It frequently happens in practice that the carrier and noise levels are known in terms of power rather than voltage. For this reason, Eq. (20-22) will be written in an alternative form using the following relations:

$$p_n = \frac{a_n^2}{R} \text{ watts/cycle} \qquad (20\text{-}23)$$

and

$$P_c = \frac{A_c^2}{2R} \text{ watts} \qquad (20\text{-}24)$$

Here, R is the circuit impedance at the point where the noise and carrier voltages are defined while p_n and P_c are the corresponding powers. Substituting in Eq. (20-22) for a_n and A_c gives the following alternative expression for the rms frequency deviation due to the two noise bands each of width δf.

$$\text{Rms frequency deviation due to two } \delta f \text{ bands of noise, one at } f_c + \omega_1/2\pi \text{ and the other at } f_c - \omega_1/2\pi = f_1 \sqrt{\frac{p_n \, \delta f}{P_c}} \text{ cps} \qquad (20\text{-}25)$$

Noise at 0 TL in an FM System

To determine the noise in dbrn at the 0-db transmission level from Eq. (20-22) or (20-25), some relationship between the rms frequency deviation of a signal in the radio system and the power of that signal at the 0 TL is needed. If the deviation sensitivity C_{FM} of the FM detector and the transmission level at the output of the detector are both known, the noise power can be determined. However, it is unlikely that either of these values will be known during the early design stages of a radio system. As an alternative, then, use can be made of the relationship between the multiplex signal which the system is to handle and the peak frequency deviation which this signal will produce. Chapter 10 discusses and shows how to determine the peak load, P_S, which an AM carrier telephone system must be designed to carry. If this load is now applied to an FM system, the peak frequency deviation produced in the system is taken to correspond to the peaks of a sinusoidal baseband signal having a power of P_S dbm at the 0 TL. The peak frequency deviation for the system is usually established early in the design, at least tentatively, so that this factor together with the value of P_S for the load to be carried will permit the system noise to be evaluated. If the peak deviation is ΔF, and the variation is sinusoidal, the rms deviation is $\Delta F/\sqrt{2}$. Let $\bar{F}$ denote this rms value, which will now be assumed to correspond to the power (or rms voltage) of P_S dbm at the 0 TL. Clearly, any rms frequency deviation equal to $\bar{F}$ will produce P_S dbm at the zero level. Conversely, the baseband power P_n in dbm at the zero level produced by any other rms frequency deviation $\bar{f}$ can be expressed as follows:

$$P_n = P_S + 20 \log \frac{\bar{f}}{\bar{F}} \qquad \text{dbm at 0 TL} \qquad (20\text{-}26)$$

where

$\bar{F}$ = rms frequency deviation produced by the baseband signal P_S

$\bar{f}$ = rms frequency deviation produced by any other signal having a power of P_n dbm at 0 TL

It follows, then, that if $\bar{f}$ represents the rms frequency deviation due to a band of noise, P_n will represent the resulting baseband noise power. For the case of noise in the top message channel, an approximate expression for $\bar{f}$ is given by Eq. (20-25). Substituting this equation in Eq. (20-26) gives

$$P_n = P_S + 20 \log \frac{f_1}{\bar{F}} \sqrt{\frac{p_n\,\delta f}{P_c}} \qquad \text{dbm at 0 TL} \qquad (20\text{-}27)$$

In practice, p_n and P_c are usually expressed in dbm/cps and dbm, respectively. Furthermore, reference is usually made to the peak frequency deviation $\Delta F = \sqrt{2}\,\bar{F}$ rather than the rms deviation $\bar{F}$. For

these reasons it is convenient to rewrite Eq. (20-27) in the form

$$P_n = P_S + 20 \log \frac{f_1}{\Delta F} + 10 \log 2\delta f + p_n \big|_{\text{dbm/cps}} - P_c \big|_{\text{dbm}} \qquad \text{dbm at 0 TL} \qquad (20\text{-}28)$$

Equation (20-27) or Eq. (20-28) gives the noise in dbm which would be measured at 0 TL in a narrowband δf cps wide due to fluctuation noise having a power density of p_n watts per cycle (or p_n dbm per cycle) at a point where the carrier power is P_c watts (or P_c dbm). The multiplexed telephone channel is located at a baseband frequency f_1, and ΔF is the peak frequency deviation of the FM system. Normally, multiplexed telephone channels are spaced at 4-kc intervals and have an effective noise bandwidth of 3 kc. Thus, δf is normally assumed to be 3 kc.

Equation (20-27) is the basis for a simple procedure which is occasionally used to determine the noise at 0 TL due to fluctuation noise in the FM portion of the system. Substituting $\Delta F = \sqrt{2}\,\bar{F}$ permits writing Eq. (20-27) as

$$P_n = P_S + 20 \log \frac{f_1}{\Delta F} + 10 \log \frac{2\delta f p_n}{P_c} \qquad \text{dbm at 0 TL} \qquad (20\text{-}29)$$

The calculation is first made for a channel located at a baseband frequency equal to the peak frequency deviation. This makes $20 \log f_1/\Delta F = 0$. The ratio of carrier power to noise power in a band $2\delta f$ then determines the quantity $10 \log P_c/2\delta f p_n$ db. The noise at 0 TL is then this number of db below P_S. Since the noise varies at 6 db per octave as a function of channel location, it is a simple matter to determine the noise in any other channel. In practice, when p_n and P_c are given in dbm/cps and dbm, respectively, Eq. (20-28) rather than Eq. (20-27) is used for this calculation.

Illustrative Example II

Problem

The preceding analysis can be illustrated by determining the noise, in dbrn at the 0 TL, in the noisiest channel at the output of an FM system due to the thermal-type noise contributed by each repeater. The system constants will be assumed to be as follows:

1. *Baseband signal:* 1000 single-sideband telephone multiplex channels. Each channel is 3 kc wide; the spacing between channels is 4 kc. The baseband signal is thus 4 mc wide and will be assumed to extend from 0 to 4 mc.

2. *Repeaters:* The system consists of 100 repeaters in tandem. Each repeater has an input carrier power of -30 dbm and a noise figure of 12 db. The repeater bandwidth is 20 mc.
3. *Peak frequency deviation:* 4 mc.

Solution

First, a check should be made to insure that the total noise power is small compared to that of the unmodulated carrier. Thermal noise at room temperature is -174 dbm/cps. The noise power in a 20-mc band is 10 log 20 $(10^6) = 73$ db higher, or -101 dbm. Increased by the 12-db noise figure, the total noise power at the input to a single repeater is -89 dbm. For 100 repeaters in tandem, the total noise power at the input to the last repeater will be 10 log 100 $= 20$ db higher, or -69 dbm. With an unmodulated carrier power of -30 dbm, the carrier-to-noise ratio at the input to the last repeater is thus 39 db, which is more than adequate to meet the requirement.

Using the methods of Chap. 10, P_S for this system is about $+24$ dbm (based on $V_o = -12.5$ vu, $\sigma = 5$ db). This signal produces a peak frequency deviation of 4 mc or an rms frequency deviation of $4/\sqrt{2} = 2\sqrt{2}$ mc.

The noisiest channel in an FM system will be the top baseband channel due to the triangular noise spectrum of the system. The noise in a 3-kc band located at 4 mc on each side of the carrier produces an rms frequency deviation of the carrier given by Eq. (20-22), repeated here for convenience:

$$\text{Rms frequency deviation} = \frac{a_n f_1 \sqrt{2\delta f}}{A_c}$$

The only values not clearly given in this problem are a_n and A_c. Since thermal noise is -174 dbm/cps and the noise figure is 12 db, the noise power at the input to a single repeater is -162 dbm/cps. The carrier at this point is -30 dbm. The peak power of a sinusoidal signal is 3 db higher than the average power; hence, the peak power of the carrier is -27 dbm. Therefore, the ratio of the noise power in a 1-cycle band to the peak carrier power at the input to one repeater is $-162 - (-27) = -135$ db. This ratio is, of course, proportional to a_n^2/A_c^2. Thus, 20 log $a_n/A_c = -135$, from which $a_n/A_c = 1.78\ (10^{-7})$. Substituting this into Eq. (20-22) and solving for the rms frequency deviation, which for convenience will be denoted by $\bar{f}$, gives

$$\begin{aligned}\bar{f} &= (1.78 \times 10^{-7})\ (4 \times 10^6)\ \sqrt{2(3 \times 10^3)} \\ &= 55.2 \text{ cps}\end{aligned}$$

From Eq. (20-26), the noise power in a 3-kc band at 4 mc due to a single repeater is thus

$$\begin{aligned}P_n &= P_S + 20 \log \frac{\bar{f}}{F} \\ &= 24 + 20 \log \frac{55.2}{(4 \times 10^6)/\sqrt{2}} \\ &= -70 \text{ dbm at 0 TL}\end{aligned}$$

For 100 repeaters, the noise power would be 20 db higher, or -50 dbm at 0 TL. This corresponds to $88 - 50 = 38$ dbrn at the zero level.

To illustrate the alternative approach using Eq. (20-28), the carrier power P_c at the input to a single repeater is -30 dbm. The random noise power p_n is -174 dbm per cycle increased by 12 db to account for the receiver noise figure; i.e., $p_n = -174 + 12 = -162$ dbm/cycle. The factor 10 log $2\delta f$ equals 37.8 db for $\delta f = 3$ kc. Thus, the ratio of the carrier power to noise power in a 6-kc band is $-30 - (-162 + 37.8) = 94$ db. Since the peak frequency deviation is 4 mc, the baseband noise P_n at 0 TL in a channel located at this frequency will be 94 db below P_S; i.e., $P_n = 24 - 94 = -70$ dbm. Since 4 mc is the location of the top channel, this will be the noisiest channel. For 100 repeaters in tandem, the noise at 0 TL will be $-70 + 10 \log 100 = -50$ dbm or 38 dbrn.

The baseband noise spectrum has a 6-db-per-octave slope. Thus, the noise in telephone channels located below 4 mc will be less. For example, the noise in a telephone channel at 2 mc would be $+32$ dbrn. The channel noise versus frequency of channel is shown by the solid line in Fig. 20-4. The dashed line will be explained in the next section.

Comparison of FM and PM System Noise

The analysis of the previous sections has shown that the random noise in a telephone channel at the output of an FM system is dependent on the frequency of the channel. Thus, if the top channel just meets requirements, the lower frequency channels have unnecessary margin. This is not very efficient.

In a PM system on the other hand, the noise is the same in all the telephone channels, since the phase modulation due to the signal and that due to the random noise are both flat with frequency. In this section, the noise in a PM system is compared with the noise in the top channel of an FM system, under the condition that the rms frequency deviation caused by the transmitted signal is the same for the two

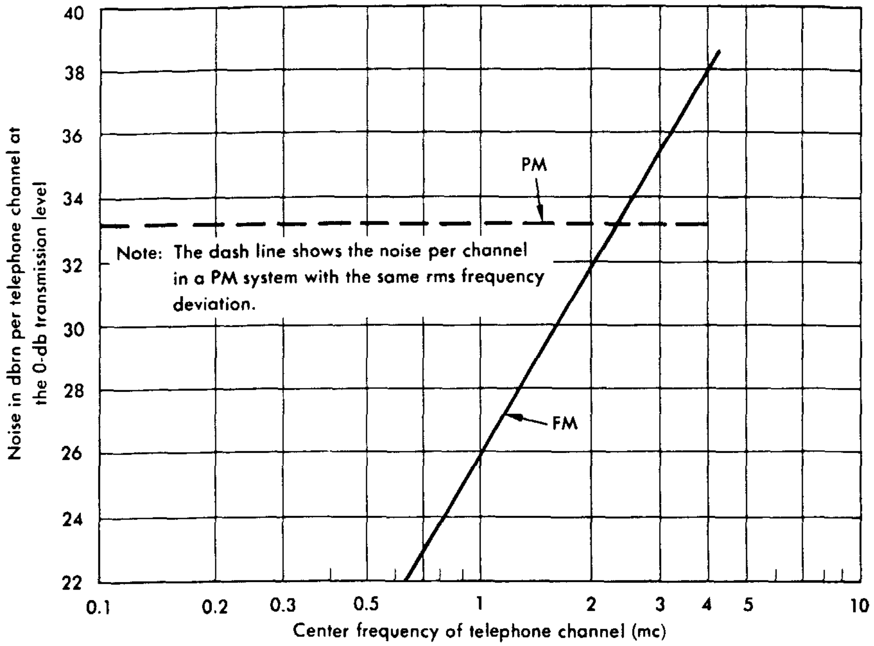

FIG. 20-4. Noise per channel at the output of the FM system, considered in Illustrative Example II.

systems. In addition, it is assumed that the power spectrum of the baseband signal is flat from 0 to f_1 cps.

An easy way to approach this problem is to compare the output of an FM modulator with the output of a PM modulator where the PM modulator consists of a differentiator and an FM modulator (see Chap. 19). This is shown in Fig. 20-5. In the first case, where an FM wave is produced, the power spectrum, S, of the signal applied

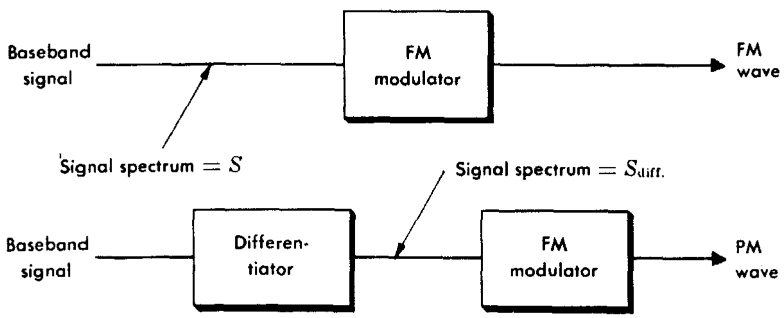

FIG. 20-5. FM and PM modulators.

to the FM modulator can be written as

$$\begin{aligned} S &= a^2 \text{ watts/cps} \qquad 0 < f < f_1 \text{ cps} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{20-30}$$

The total baseband power applied to the FM modulator is

$$\text{FM power} = a^2 f_1 \text{ watts} \tag{20-31}$$

In the second case, where a PM wave is produced, the power spectrum S_{diff} of the signal applied to the FM modulator is parabolic. This is because the voltage spectrum at the output of the differentiator is proportional to the frequency (e.g., if the input is $A_v \sin \omega_v t$, the output is $A_v\omega_v \cos \omega_v t$), and the power spectrum is proportional to the square of the voltage spectrum; that is, $S_{\text{diff}} = kf^2$, where k is a constant. If, for the moment, the power spectrum in the top channel at the differentiator output is set equal to that in the FM case (a^2), it follows that $k = a^2/f_1^2$ and the expression for S_{diff} becomes

$$\begin{aligned} S_{\text{diff}} &= \frac{a^2}{f_1^2} f^2 \text{ watts/cps} \qquad 0 < f < f_1 \text{ cps} \\ &= 0 \text{ elsewhere} \end{aligned} \tag{20-32}$$

From this, the total power applied to the FM modulator is

$$\begin{aligned} \text{PM power} &= \int_0^{f_1} \frac{a^2}{f_1^2} f^2 \, df \\ &= \frac{1}{3} a^2 f_1 \text{ watts} \end{aligned} \tag{20-33}$$

The signal levels in the top channel have been set equal at the FM modulator input for both cases. The frequency (or phase) deviation caused by the top channel signal will be the same, therefore, in the two systems, and the top channel S/N ratios in the radio links of the two systems will be identical. At baseband output, the 0 TL noise in the top channels of the two systems must therefore be the same.

However, the total power applied to the FM modulator which is producing phase modulation is only one-third as great as that applied to the other modulator. Therefore, the rms frequency deviation is less by a factor of $\sqrt{3}$. To make the rms deviation the same for both cases, the signal level in the PM system must be raised by $20 \log \sqrt{3} = 4.8$ db. Thus for the same rms frequency deviation, there is a 4.8-db signal-to-noise advantage for pure PM over pure FM. In other words, for the same rms frequency deviation, the random noise in each of the channels at the output of a PM system is 4.8 db below the noise in the worst channel at the output of an FM system. This advantage is shown by the dashed line in Fig. 20-4 for the illustrative problem of the previous section.

Pre-emphasis and De-emphasis

Closer examination of the use of a differentiator and an FM modulator to obtain phase modulation reveals a problem, illustrated in Fig. 20-6. The signal output of the differentiator is triangular, as shown. However, thermal noise which exists in the circuit at the output of the differentiator has a spectrum which, for purposes of this example, will be assumed to be flat with frequency. (At low frequencies, for example, the noise may actually increase with decreasing frequency, perhaps at a $1/f$ rate.) At the lowest frequencies of interest, the noise power may exceed the signal power. Therefore, it is impractical to provide pure phase modulation of the baseband signal by this method. Some improvement in the over-all noise performance of the system can still be obtained, however, if the differentiator is modified to provide phase modulation only at the higher frequencies and frequency modulation at the lower frequencies. This approach to improving the system signal-to-noise performance is generally referred to as pre-emphasis.

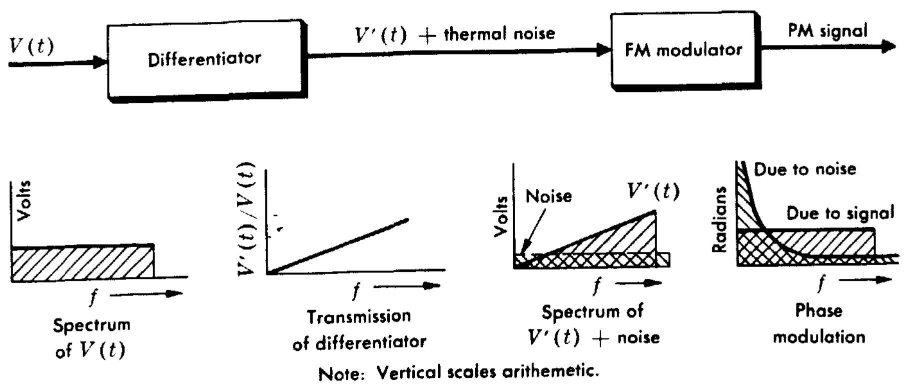

FIG. 20-6. Noise problem when FM modulator and differentiator are used to obtain phase modulation.

In general, pre-emphasis is produced by passing the baseband signal through an appropriate network before the signal is applied to the FM modulator. Thus, as illustrated in Fig. 20-6, a differentiator is actually a form of a pre-emphasis network which shapes a flat spectrum so that it has a triangular or +6-db-per-octave envelope.

A pre-emphasis shape which would avoid the noise problem of Fig. 20-6 is illustrated in Fig. 20-7. Here, shaping is done only at the higher baseband frequencies, resulting in phase modulation at these frequencies and frequency modulation at the lower frequencies. When pre-emphasis is used ahead of the FM modulator, the baseband signal at the output of an FM demodulator, at the receiving end of the system, will have the same shape as the pre-emphasized signal. A de-emphasis network, having a characteristic opposite to that of the pre-emphasis network, is connected to the output of the FM demodulator to restore the baseband signal to its normal shape.

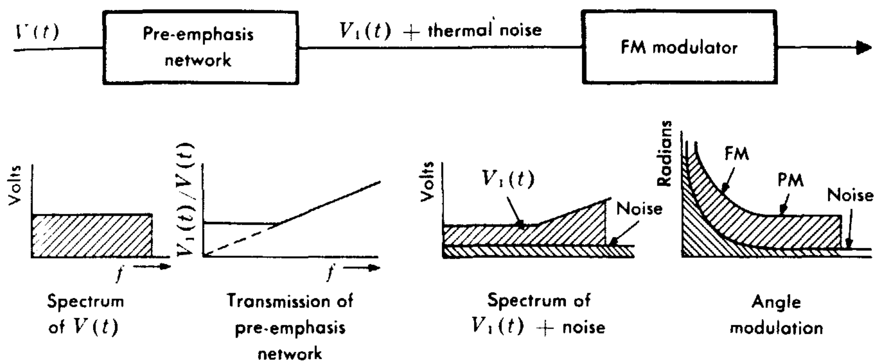

FIG. 20-7. Pre-emphasis with FM modulator.

Broadcast FM uses a pre-emphasized signal similar to that illustrated in Fig. 20-7. De-emphasis is provided in the individual home receivers. It should be noted that any system which is neither pure FM nor pure PM is usually referred to as an FM system. In fact, it is rather common practice to use the term frequency modulation to denote any form of angle modulation.

The 4.8-db advantage of PM over FM derived in the previous section is the maximum advantage which can be obtained by pre-emphasis in front of an FM modulator. Since noise prevents the use of pure PM, only about 3 to 4 db of improvement in the top channel noise can actually be achieved in practice through the use of suitably shaped pre-emphasis. Figure 20-8 shows the transmission characteristic of the pre-emphasis network used in TD-2 for telephone transmission. When this network is introduced in front of the FM modulator, it is apparent that the rms deviation will be reduced. In practice, the introduction of the network is accompanied by an increase in gain of the amplifier preceding the modulator which restores the rms deviation to approximately its unpre-emphasized value. The combination of pre-emphasis network and increased amplifier gain results in signals in the neighborhood of 1 mc being unaffected by the introduction of the pre-emphasis network. Lower frequency signals are attenuated while higher frequency signals undergo amplification. In TH, stepped pre-emphasis is used, in which each 600-channel mastergroup has a different transmission level at the input to the FM terminal transmitter. Mastergroup 2 is transmitted about 5 db higher and mastergroup 3 about 7.5 db higher than mastergroup 1. This arrangement avoids the use of special networks, but only about 2.5 db of improvement in the top channel noise is achieved.

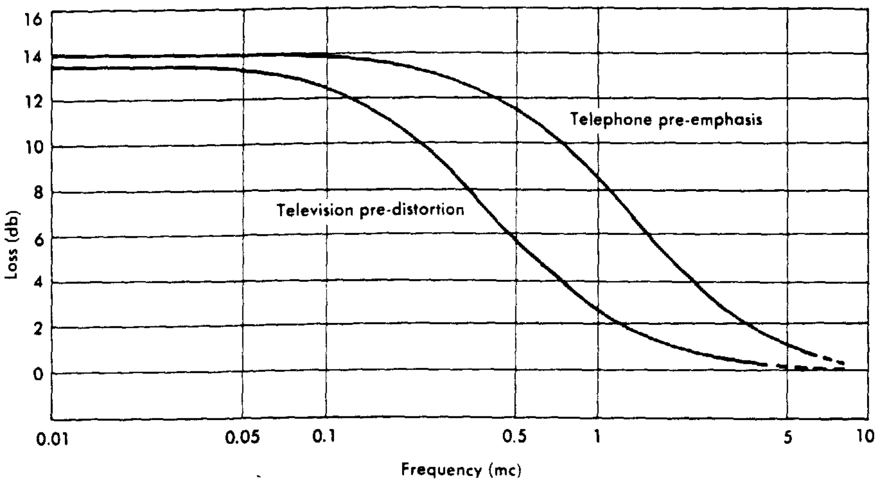

FIG. 20-8. Typical pre-emphasis network characteristics.

Figure 20-8 also shows the baseband shaping used in TD-2 for television transmission. For TV transmission, the baseband shaping is generally referred to as predistortion rather than pre-emphasis. As discussed in Chap. 16, a TV signal is more tolerant of high-frequency noise than low-frequency, so that pre-emphasis, as it applies to improving the noise performance at the high end of the baseband, is not required. Instead, the signal is predistorted ahead of the FM terminal transmitter to improve the transmission of the color information. In practice, a given channel may at any time have to carry either a black and white or a color TV signal. Because of this, predistortion is always used even though it may not be essential for

black and white transmission. There are two ways of viewing the manner in which predistortion helps the transmission of a color signal.

First, from the quasi-stationary viewpoint, the exact frequencies at which the 3.58-mc color subcarrier and its sidebands are being transmitted through the system vary according to the amount of grayness in the picture, which is changing at a relatively slow rate. These changes in transmission frequencies result in variations of the amplitude and phase (called differential gain and phase) of the 3.58-mc color carrier, which affects the hue and intensity of the colors. Predistortion, by reducing the amplitude of the low-frequency components of the TV signal, reduces the excursions of the 3.58-mc carrier and accordingly reduces the differential gain and phase.

From a second, and more precise, viewpoint, the transmission distortions of the system produce intermodulation products between the color carrier and the low-frequency components of the TV signal. (The effects of transmission distortions are discussed in Chap. 21.) These intermodulation products fall in the color band and affect the color transmission. Reducing the amplitude of the low-frequency components reduces the magnitude of the modulation products in which they are involved and thereby improves the color performance.

FM Advantage

It is possible to get better signal-to-noise performance in a frequency modulation system than in an AM system with the same transmitted power. To achieve this advantage, however, it is necessary to use large indices of modulation. Higher order sidebands become important, and a wider bandwidth is required than would be necessary for the corresponding AM system. The improvement in the signal-to-noise performance which is obtained by using wider bandwidths is sometimes referred to as the FM advantage. To examine this quantitatively, a single-sideband AM system with suppressed carrier will be compared with an FM system. The peak power is assumed to be the same for both systems.

In the AM system, let

P_S = power in dbm, at 0-db transmission level, of the largest sine wave that the system is designed to transmit.

N = noise in dbm in a 3-kc band at some low-level point (for example, a repeater input).

P_{SL} = power in dbm of the largest sine wave that the system is designed to transmit, at the low-level point in the system. (Hence, $P_S - P_{SL}$ is the gain in the system between the low-level point and zero transmission level.)

The noise in a 3-kc telephone channel at 0-db transmission level can then be written as

$$\text{Noise}_{\text{AM}} = P_S - P_{SL} + N \qquad \text{dbm} \tag{20-34}$$

In the FM system, two additional quantities are defined. These are the peak frequency deviation and the center frequency of the top telephone channel, where the noise will be the highest. Thus,

ΔF = peak frequency deviation in cps.
f_1 = center frequency of top channel in cps.

The rms frequency deviation due to the noise centered at $f_c + f_1$ and $f_c - f_1$ is given by Eq. (20-22). Let $\bar{f}$ denote this rms deviation. Thus,

$$\bar{f} = \frac{a_n f_1 \sqrt{2\delta f}}{A_c} \tag{20-35}$$

This rms frequency deviation can be related to the AM system parameters, previously defined, as follows. Taking 20 log of both sides of Eq. (20-35) gives

$$\begin{aligned} 20 \log \bar{f} &= 20 \log a_n \sqrt{2\delta f} - 20 \log A_c + 20 \log f_1 \\ &= 10 \log (a_n \sqrt{2\delta f})^2 - 10 \log A_c{}^2 + 20 \log f_1 \end{aligned} \tag{20-36}$$

The term 10 log $(a_n \sqrt{2\delta f})^2$ represents the noise power in two δf bandwidths due to the noise voltage a_n volts/cps acting across the system impedance. Note that because of the ratio, a_n/A_c, the impedance cancels and is therefore not important in this problem. If δf is 3 kc, it follows that at the low-level point in the system, the term 10 log $(a_n \sqrt{2\delta f})^2$ can be written as equal to $N + 3$ dbm, the 3 db being due to the presence of two 3-kc bands. In a similar manner, the term 10 log $A_c{}^2$, which represents the peak carrier power, can be written as equal to $P_{SL} + 3$ dbm. This follows from the assumption that the peak power in the two systems is the same; the 3 db relates the average power at the low-level point to the peak power at that point. Thus, Eq. (20-36) can be written as

$$\begin{aligned} 20 \log \bar{f} &= (N + 3) - (P_{SL} + 3) + 20 \log f_1 \\ &= N - P_{SL} + 20 \log f_1 \end{aligned} \tag{20-37}$$

The rms frequency deviation $\bar{f}$ will produce a baseband noise power which can be determined from ΔF and P_S, using Eq. (20-26). Let P_n represent the FM system noise power in the top 3-kc channel; thus, Eq. (20-26) can be written as

$$\text{Noise}_{\text{FM}} = P_S + 20 \log \bar{f} - 20 \log \bar{F} \tag{20-38}$$

Use of $\bar{F} = \Delta F/\sqrt{2}$ and Eq. (20-37) reduces Eq. (20-38) to

$$\text{Noise}_{\text{FM}} = P_S - P_{SL} + N - 20 \log \frac{\Delta F}{f_1\sqrt{2}} \tag{20-39}$$

Substituting Eq. (20-34) gives

$$\text{Noise}_{\text{FM}} = \text{noise}_{\text{AM}} - 20 \log \frac{\Delta F}{f_1\sqrt{2}} \tag{20-40}$$

Equation (20-40) shows that the FM advantage is

$$\text{FM advantage} = 20 \log \frac{\Delta F}{f_1\sqrt{2}} \tag{20-41}$$

Unless the peak frequency deviation is equal to or greater than the $\sqrt{2}$ times the frequency of the top telephone channel, the FM advantage is negative. For an FM system where the peak frequency deviation is equal to the frequency of the top transmitted channel, the FM advantage is -3 db, and the noise in the top channel would be 3 db higher than in a single-sideband AM system. With pre-emphasis, the FM advantage can be raised to about 0 db when the peak frequency deviation is equal to the frequency of the top channel.

By the approximate rule due to Carson mentioned in Chap. 19, the bandwidth required to transmit an FM (or PM) signal is twice the sum of the peak deviation and the top baseband frequency. When these are equal, the required bandwidth is therefore four times the top frequency. Since a single-sideband AM system can be transmitted in a bandwidth equal to the top frequency, the pre-emphasized FM system requires at least four times the bandwidth of the AM system to achieve the same noise performance.

The apparent advantage of AM from the point of view of bandwidth utilization would appear to favor this form of modulation in microwave radio systems. As discussed in Chap. 17, the practical problems associated with obtaining linear amplifiers with adequate gain and power output make the use of AM impractical at the present time.

Interference and Random Noise in Large-Index Systems

The analysis of the preceding sections assumes that the carrier is essentially unmodulated, or, equivalently, that the system is low-index with most of the power in the carrier. When the signal modulation on the carrier cannot be neglected, a more complicated equation, Eq. (20A-18) of the appendix, applies. The phase distortion (or instantaneous phase deviation) due to the interference is, from Eq. (20A-18),

$$\text{Phase distortion} = \sum_{n=1}^{N} \frac{A_n(t)}{A_c} \sin\left[\omega_n t + \theta_n(t) - \phi(t)\right] \qquad (20\text{-}42)$$

If $A_n(t) = A_n$, and $\theta_n(t) = \theta_n$ (i.e., if both are constants), Eq. (20-42) becomes

$$\text{Phase distortion} = \sum_{n=1}^{N} \frac{A_n}{A_c} \sin\left[\omega_n t + \theta_n - \phi(t)\right] \qquad (20\text{-}43)$$

In these equations, $\phi(t)$ represents the desired phase modulation of the carrier by the baseband signal.

Equation (20-43) shows that each of the noise components is actually an FM wave with the same index of modulation as the original signal. Therefore, if the sideband energy of the transmitted FM wave is small, the sideband about each noise component will also be small.

The frequency modulation produced by the noise is obtained by taking the derivative of Eq. (20-43). Thus,

$$\text{Frequency deviation} = \frac{d}{dt} \sum_{n=1}^{N} \frac{A_n}{A_c} \sin\left[\omega_n t + \theta_n - \phi(t)\right]$$

$$= \sum_{n=1}^{N} \frac{A_n}{A_c} \left[\omega_n - \phi'(t)\right] \cos\left[\omega_n t + \theta_n - \phi(t)\right] \qquad (20\text{-}44)$$

For high-index systems where the number of random noise components is small, the expression above may lead to results which are quite different from those obtained for a low-index system. However, when the interference consists of many components, such as is the case with random noise, the spectrum of the phase distortion term is usually relatively flat with frequency even though it consists of a large number of small FM waves instead of a large number of individual noise components. The frequency deviation obtained by taking the derivative therefore tends to have a triangular spectrum which, for all practical purposes, is the same as the one obtained for a low-index signal.

Breaking Region

The previous sections have treated the phase and frequency modulation which is produced by random noise when the total noise power is much less than the carrier power. As long as this is the case, the signal-to-noise ratio in the baseband output varies linearly with the signal-to-noise ratio in the FM or PM portion of the system. When the carrier power is less than about ten times the noise power, this linearity no longer holds and the output signal-to-noise ratio decreases faster than the input signal-to-noise ratio decreases. In this region, which is referred to as the breaking region, the system rapidly becomes unusable.

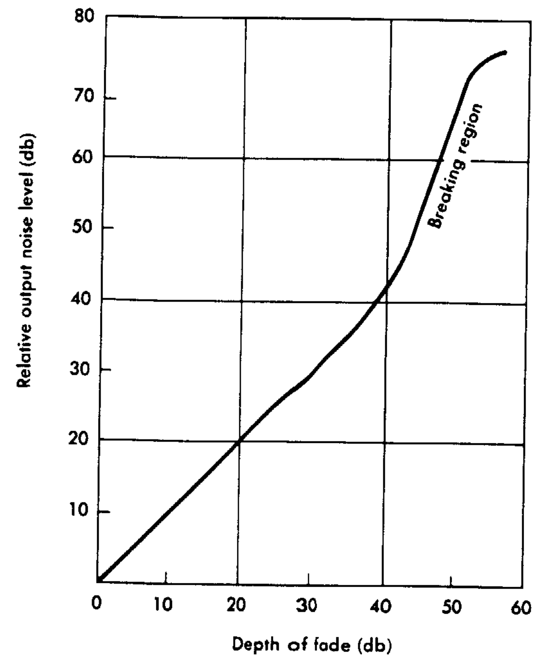

FIG. 20-9. Typical effect of breaking region on output noise during deep fades.

This effect is illustrated in Fig. 20-9 for a system in which the output signal is kept constant during the fade by an automatic gain control. Consequently, the gain increases as the radio signal fades, and the output noise increases linearly with the depth of fade until the breaking region is reached. The noise then increases at a faster rate.

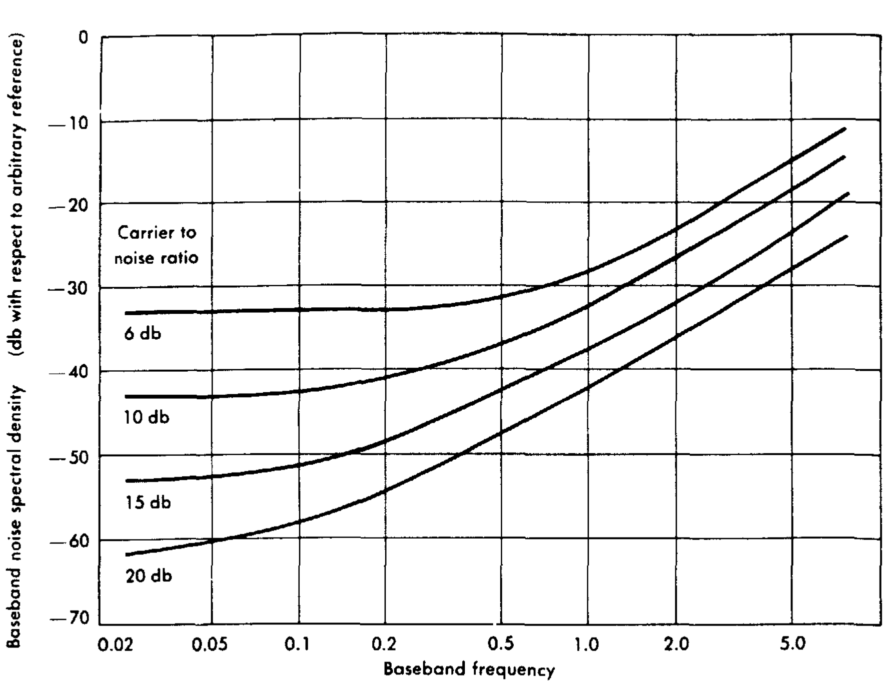

FIG. 20-10. Baseband noise versus frequency.

The onset of breaking as it affects noise appears first at low-baseband frequencies as illustrated in Fig. 20-10. If this noise is monitored on headphones, it will manifest itself as intermittent "clicks" which become more frequent as the breaking region is approached. Beyond the breaking region, as the carrier-to-noise ratio is decreased, the clicks rapidly merge into a crackling or sputtering sound.

An examination of Fig. 20-10 shows that the noise which appears in the breaking region has a flat spectral density and is superimposed on the normal triangular FM noise spectrum. To illustrate qualitatively how this additional flat noise is generated, reference will be made to the vector diagram of Fig. 20-11. For small amounts of noise, the tip of the resultant vector will normally reside in the neighborhood of the point Q. When the carrier-to-noise ratio is reduced to a value of about 12 db or less, the noise vector will occasionally exceed the carrier amplitude as illustrated by the vector OP'. If the direction of the noise vector is such that the tip of the resultant P' is in the neighborhood of the origin O, this resultant vector may undergo sudden phase jumps of π or 2π radians. This is illustrated in Fig. 20-12(a) which shows the phase of the resultant as a function of time. Since the baseband output of the FM system is proportional to the instantaneous frequency or the rate of change of phase, the phase steps of Fig. 20-12(a) correspond to frequency spikes as illustrated in Fig. 20-12(b). It can be shown that the spikes resulting from the 2π jumps have considerably more low-frequency content than the π jumps and are therefore the major noise contributors. These spikes give rise to the audible clicks mentioned previously and have a spectral density at baseband which is approximately flat. As the carrier to noise ratio decreases, the number of spikes per second will increase, giving rise to the growing flat portion of the spectrum in Fig. 20-10.

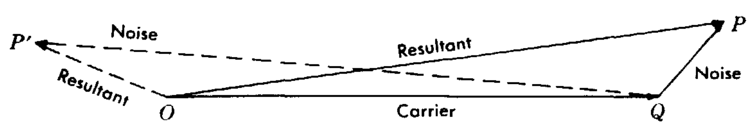

FIG. 20-11. Illustration of effect of noise on carrier phase.

Quantitative methods for determining the amount of baseband noise in the breaking region are discussed in References 3 and 4.

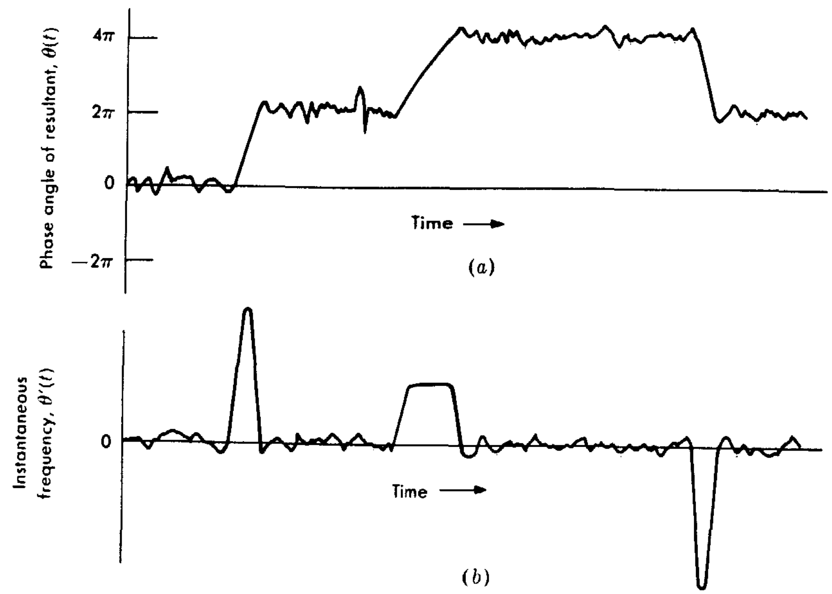

FIG. 20-12. Phase angle and instantaneous frequency of resultant vector shown in Fig. 20-11.

APPENDIX

In this appendix, the effect of an interfering signal on an FM signal is derived for a completely general case. Approximate expressions for special cases are then obtained by appropriate substitutions.

Let the FM signal be given by

$$S(t) = A_c \cos [\omega_c t + \phi(t)] \tag{20A-1}$$

where

A_c = peak carrier amplitude
ω_c = carrier frequency, in radians/sec
$\phi(t)$ = angle modulation in radians

The interfering signal can be expressed as

$$I(t) = A_n(t) \cos [(\omega_c + \omega_n)t + \theta_n(t)] \tag{20A-2}$$

where

A_n = peak amplitude of the interference
$A_n(t)$ = amplitude modulation of the interference
$\omega_c + \omega_n$ = frequency of interference, in radians/sec
$\theta_n(t)$ = angle modulation of the interference

The combination of the desired FM signal and the interference is

$$\begin{aligned} M(t) &= S(t) + I(t) \\ &= A_c \cos [\omega_c t + \phi(t)] \\ &\qquad + A_n(t) \cos [(\omega_c + \omega_n)t + \theta_n(t)] \end{aligned} \tag{20A-3}$$

Equation (20A-3) can be written as

$$\begin{aligned} M(t) &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t)]} + A_n(t)\, e^{j[(\omega_c + \omega_n)t + \theta_n(t)]} \right) \\ &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t)]} \right) \left(1 + \frac{A_n(t)}{A_c} e^{j[\omega_n t + \theta_n(t) - \phi(t)]} \right) \end{aligned} \tag{20A-4}$$

For convenience define

$$\rho = \omega_n t + \theta_n(t) - \phi(t) \tag{20A-5}$$

$$b = \frac{A_n(t)}{A_c} \tag{20A-6}$$

Substituting Eqs. (20A-5) and (20A-6) into Eq. (20A-4) gives

$$\begin{aligned} M(t) &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t)]} \right) \left(1 + b e^{j\rho} \right) \\ &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t)]} \right) (1 + b \cos \rho + jb \sin \rho) \end{aligned} \tag{20A-7}$$

If

$$\Phi = \tan^{-1} \frac{b \sin \rho}{1 + b \cos \rho} \tag{20A-8}$$

Eq. (20A-7) can be written as

$$\begin{aligned} M(t) &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t)]} \right) \left(\sqrt{(1 + b \cos \rho)^2 + (b \sin \rho)^2}\; e^{j\Phi} \right) \\ &= \begin{bmatrix} \text{Real} \\ \text{part} \\ \text{of} \end{bmatrix} \left(A_c\, e^{j[\omega_c t + \phi(t) + \Phi]} \right) \left(\sqrt{1 + 2b \cos \rho + b^2} \right) \\ &= A_c \sqrt{1 + 2b \cos \rho + b^2} \cos [\omega_c t + \phi(t) + \Phi] \end{aligned} \tag{20A-9}$$

Equation (20A-9) is the general result and indicates the amplitude and phase modulation of the desired signal that is produced by the interference.

By considerable trigonometric manipulation, not given here, the series expansion for Φ to third order terms can be shown to be

$$\Phi = b \sin \rho - \frac{b^2}{2} \sin 2\rho + \frac{b^3}{3} \sin 3\rho - \cdots \tag{20A-10}$$

If $b = A_n(t)/A_c$ is small, that is, if, for example,

$$\text{Maximum } |b| < 0.1, \tag{20A-11}$$

the higher-order terms of Eq. (20A-10) can be dropped. In addition, the amplitude term of Eq. (20A-9) can be written as

$$\sqrt{1 + 2b \cos \rho + b^2} \cong 1 + b \cos \rho \qquad \text{(20A-12)}$$

Thus, for this condition, Eq. (20A-9) reduces to

$$\begin{aligned} M(t) &\cong A_c [1 + b \cos \rho] \cos [\omega_c t + \phi(t) + b \sin \rho] \\ &\cong A_c \left\{ 1 + \frac{A_n(t)}{A_c} \cos [\omega_n t + \theta_n(t) - \phi(t)] \right\} \\ &\quad \cdot \cos \left\{ \omega_c t + \phi(t) + \frac{A_n(t)}{A_c} \sin [\omega_n t + \theta_n(t) - \phi(t)] \right\} \end{aligned} \qquad \text{(20A-13)}$$

If

$A_n(t) = A_n$ (i.e., a constant)
$\phi(t) = 0$ (i.e., the desired carrier is unmodulated)
$\theta_n(t) = \theta_n$ (i.e., a constant),

Eq. (20A-13) becomes

$$\begin{aligned} M(t) &\cong A_c \left[1 + \frac{A_n}{A_c} \cos (\omega_n t + \theta_n) \right] \\ &\quad \cdot \cos \left[\omega_c t + \frac{A_n}{A_c} \sin (\omega_n t + \theta_n) \right] \end{aligned} \qquad \text{(20A-14)}$$

which is the same as Eq. (20-2).

When there are a number of interferences, Eq. (20A-2) can be written as

$$I(t) = \sum_{n=1}^{N} A_n(t) \cos [(\omega_c + \omega_n(t) + \theta_n(t)] \qquad \text{(20A-15)}$$

The equations following (20A-2) are modified accordingly. If b in Eq. (20A-6) is replaced by b_n, i.e., if

$$b_n = \frac{A_n(t)}{A_c} \qquad \text{(20A-16)}$$

and

$$\text{Maximum} \sum_{n=1}^{N} (b_n)^2 < 0.01, \qquad \text{(20A-17)}$$

Eq. (20A-13) becomes

$$\begin{aligned} M(t) &\cong A_c \left\{ 1 + \sum_{n=1}^{N} \frac{A_n(t)}{A_c} \cos [\omega_n t + \theta_n(t) - \phi(t)] \right\} \\ &\quad \cdot \cos \left\{ \omega_c t + \phi(t) + \sum_{n=1}^{N} \frac{A_n(t)}{A_c} \sin [\omega_n t + \theta_n(t) - \phi(t)] \right\} \end{aligned} \qquad \text{(20A-18)}$$

and Eq. (20A-14) becomes

$$\begin{aligned} M(t) &\cong A_c \left[1 + \sum_{n=1}^{N} \frac{A_n}{A_c} \cos (\omega_n t + \theta_n) \right] \\ &\quad \cdot \cos \left[\omega_c t + \sum_{n=1}^{N} \frac{A_n}{A_c} \sin (\omega_n t + \theta_n) \right] \end{aligned} \qquad \text{(20A-19)}$$

Equation (20A-19) can be written in the form

$$M(t) = A_s(t) \cos [\omega_c t + \phi_s(t)] \qquad \text{(20A-20)}$$

where

$$A_s(t) \cong A_c \left[1 + \sum_{n=1}^{N} \frac{A_n}{A_c} \cos (\omega_n t + \theta_n) \right] \qquad \text{(20A-21)}$$

$$\phi_s(t) \cong \sum_{n=1}^{N} \frac{A_n}{A_c} \sin (\omega_n t + \theta_n) \qquad \text{(20A-22)}$$

Equations (20A-20), (20A-21), and (20A-22) are identical to Eqs. (20-12), (20-13), and (20-14).

In Eq. (20A-13), the frequency of the interference at baseband is obtained from the expression $\rho = \omega_n t + \theta_n(t) - \phi(t)$. When the interference is a pure sinusoid [i.e., when $\theta_n(t) = \theta_n$], and when there is no modulation on the desired carrier [i.e., $\phi(t) = 0$], the interference produces a steady tone at baseband frequency ω_n rad/sec. However, in any practical situation, the desired carrier is always modulated, even if only with unavoidable residual FM from noise (in particular, power supply harmonics) in the FM terminal transmitter. As a result, the baseband interference due to a fixed interfering tone will be frequency-modulated; that is, a steady baseband tone will not result. This is of considerable importance in tone interference analysis (Chap. 22) and when relatively narrowband selective analyzers are used in interference measurements of actual systems.

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